Chapter Ⅲ

Elementary Functions

In this chapter, we will generalize various elementary functions to corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable z that reduce to the elementary functions in calculus when $z = x + i0$. We start by defining the complex exponential function and then use it to develop the others.

§3.1. The Exponential Function

1.Properties of exponential functions

If we write

$$z_1 = x_1 + iy_1 \text{ and } z_2 = x_2 + iy_2$$

then

$$e^{z_1}e^{z_2} = (e^{x_1}e^{iy_1})(e^{x_2}e^{iy_2}) = (e^{x_1}e^{x_2})(e^{iy_1}e^{iy_2}) = e^{x_1+x_2}e^{i(y_1+y_2)} = e^{z_1+z_2}.$$

$$e^{z_1-z_2}e^{z_2} = e^{z_1} \text{ or } \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}.$$

$$\frac{d}{dz}e^z = e^z$$

If $e^z = \beta e^{i\phi}$ where $\beta = e^x$ and $\phi = y$,then $|e^z| = e^x$ and

$$\text{Arg}(e^z) = y + 2n\pi\ (n = 0, \pm1, \pm2, \ldots).$$

2. Example

Example 1. There are values of z such that

$$e^z = -1. \tag{3.1.9}$$

To find them, we write equation (3.1.9) as $e^x e^{iy} = 1e^{i\pi}$. Then, by Proposition 1.8.1, we have

$$e^x = 1 \text{ and } y = \pi + 2n\pi \quad (n = 0, \pm1, \pm2, \ldots).$$

Thus, $x = 0$, and we find that