§5.5. Examples

The coefficients in Laurent series are generally found by means other than by appealing directly to their integral representation. This is illustrated in the examples below, where it is always assumed that, when the annular domain is specified, a Laurent series for a given function is unique. As was the case with Taylor series, we defer the proof of such uniqueness until Sec. 5.8.

Example 1. Replacing z by $1/z$ in the Maclaurin series expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we gave the Laurent series representation of the function $f(z) = e^{1/z}$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{1! z} + \frac{1}{2! z^2} + \frac{1}{3! z^3} + \cdots \qquad (0 < |z| < \infty).$$

Note that no positive powers of z appear here, the coefficients of the positive powers being zero. Note, too, that the coefficient of $1/z$ is unity; and, according to Laurent's theorem in Sec. 5.4, that coefficient is the number

$$b_1 = \frac{1}{2\pi !} \int_C e^{1/z} dz,$$

where C is any positively oriented simple closed path around the origin. Since $b_1 = 1$, then,

$$\int_C e^{1/z} dz = 2\pi i.$$

This method of evaluating certain integrals around simple closed paths will be developed in considerable detail in Chap. 6.

Example 2. The function $f(z) = 1/(z-i)^2$ is already in the form of a Laurent series, where $z_0 = i$. That is,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-i)^n \qquad (0 < |z-i| < \infty),$$

where $c_{-2} = 1$ and all of the other coefficients are zero. From formula (5.4.5), Sec. 5.4, for the coefficients in a Laurent series, we know that

$$c_n = \frac{1}{2\pi i} \int_C \frac{dz}{(z-i)^{n+3}} \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

where C is, for instance, any positively oriented circle $|z-i| = R$ about the point $z_0 = i$. Thus (compare Exercise 10, Sec. 4.5)

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0, & \text{when } n \neq -2, \\ 2\pi i, & \text{when } n = -2. \end{cases}$$

Example 3. The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}, \qquad (5.5.1)$$

which has the two singular points $z = 1$ and $z = 2$, is analytic in the domains

$$|z| < 1,\ 1 < |z| < 2, \text{ and } 2 < |z| < \infty.$$

In each of those domains, denoted by D_1, D_2, and D_3, respectively, in Fig. 5-5, $f(z)$ has series representations in powers of z. They can all be found by recalling from Example 4, Sec. 5.3, that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1).$$

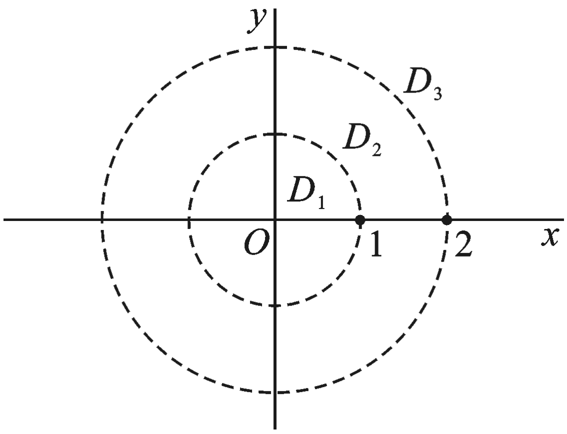

Fig. 5-5

The representation in D_1 is a Maclaurin series. To find it, we write

$$f(z) = \frac{1}{1-z} + \frac{1}{2} \cdot \frac{1}{1-(z/2)}$$

and observe that, since $|z| < 1$ and $|z/2| < 1$ in D_1,

$$f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad (|z| < 1). \tag{5.5.2}$$

As for the representation in D_2, we write

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-(1/z)} + \frac{1}{2} \cdot \frac{1}{1-(z/2)}.$$

Since $|1/z| < 1$ and $|z/2| < 1$ when $1 < |z| < 2$, it follows that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad (1 < |z| < 2).$$

If we replace the index of summation n in the first of these series by $n-1$ and then interchange the two series, we arrive at an expansion having the same form as the one in the statement of Laurent's theorem (Sec. 5.4):

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n} \quad (1 < |z| < 2). \tag{5.5.3}$$

Since there is only one such representation for $f(z)$ in the annulus $1 < |z| < 2$, expansion (5.5.3) is, in fact, the Laurent series for $f(z)$ there.

To obtain the representation of $f(z)$ in the unbounded domain D_3, we put expression (5.5.1) in the form

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-(1/z)} - \frac{1}{z} \cdot \frac{1}{1-(2/z)}$$

and observe that $|1/z| < 1$ and $1 < |z| < 2$ when $2 < |z| < \infty$, we find that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \quad (2 < |z| < \infty).$$