§3.7. Hyperbolic Functions

1.Definitions of hyperbolic functions

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as t

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}. \qquad (3.7.1)$$

2. Derivatives of hyperbolic functions

$$\frac{d}{dz}\sinh z = \cosh z, \quad \frac{d}{dz}\cosh z = \sinh z, \qquad (3.7.2)$$

3.Useful equations

$$-i\sinh(iz) = \sin z, \quad \cosh(iz) = \cos z, \qquad (3.7.3)$$

$$-i\sin(iz) = \sinh z, \quad \cos(iz) = \cosh z. \qquad (3.7.4)$$

$$\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z. \qquad (3.7.5)$$

$$\cosh^2 z - \sinh^2 z = 1, \qquad (3.7.6)$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2, \qquad (3.7.7)$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \qquad (3.7.8)$$

$$\sinh z = \sinh x \cos y + i\cosh x \sin y, \qquad (3.7.9)$$

$$\cosh z = \cosh x \cos y + i\sinh x \sin y, \qquad (3.7.10)$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y, \qquad (3.7.11)$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y, \qquad (3.7.12)$$

where $z = x + iy$.

$$|\sinh z|^2 = |\sin(-y + ix)|^2, \qquad (3.7.13)$$

where $z = x + iy$.

$$|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y.$$

$\sinh z = 0$ if and only if $z = n\pi i (n = 0, \pm 1, \pm 2, \ldots)$

$\cosh z = 0$ if and only if $z = (\frac{\pi}{2} + n\pi)i \ (n = 0, \pm 1, \pm 2, \ldots)$.

4.Definition of other hyperbolic function

$$\tanh z = \frac{\sinh z}{\cosh z} \qquad (3.7.16)$$

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z, \quad \frac{d}{dz}\cosh z = -\operatorname{csch}^2 z, \qquad (3.7.17)$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \quad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z. \qquad (3.7.18)$$

§3.8. Inverse Trigonometric and Hyperbolic Functions

1.Definitions of inverse trigonometric functions

$$\sin^{-1} z = -i\operatorname{Log}[iz + (1 - z^2)^{1/2}]. \qquad (3.8.2)$$

z.

Example. Expression (3.8.2) tells us that

$$\sin^{-1}(-i) = -i\text{Log}(1 \pm \sqrt{2}).$$

But

$$\text{Log}(1+\sqrt{2}) = \ln(1+\sqrt{2}) + 2n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\text{Log}(1-\sqrt{2}) = \ln(\sqrt{2}-1) + (2n+1)\pi i \ (n = 0, \pm 1, \pm 2, \ldots).$$

Since $\ln(\sqrt{2}-1) = \ln\dfrac{1}{1+\sqrt{2}} = -\ln(1+\sqrt{2})$,

Then, the numbers

$$(-1)^n \ln(1+\sqrt{2}) + n\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$$

constitute the values of $\text{Log}(1 \pm \sqrt{2})$. Thus, in rectangular form,

$$\sin^{-1}(-i) = n\pi + i(-1)^{n+1} \ln(1+\sqrt{2}) \ (n = 0, \pm 1, \pm 2, \ldots).$$

$$\cos^{-1} z = -i\text{Log}[z + i(1-z^2)^{1/2}] \tag{3.8.3}$$

$$\tan^{-1} z = \frac{i}{2}\text{Log}\frac{i+z}{i-z}. \tag{3.8.4}$$

2. Derivatives of inverse trigonometric functions

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}},$$

$$\frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}.$$

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2},$$

3. Derivatives of inverse hyperbolic functions

$$\sinh^{-1} z = \text{Log}[z + (z^2+1)^{1/2}],$$

$$\cosh^{-1} z = \text{Log}[z + (z^2-1)^{1/2}]$$

$$\tanh^{-1} z = \frac{1}{2}\text{Log}\frac{1+z}{1-z}.$$