§4.15. Maximum Modulus Principle

In this section, we derive an important result involving maximum values of the moduli of analytic functions. We begin with a needed lemma.

Lemma 4.15.1. *Suppose that* $|f(z)| \le |f(z_0)|$ *at each point* z *in some neighborhood* $|z - z_0| < \varepsilon$ *in which* f *is analytic. Then* $f(z) = f(z_0)$ *throughout that neighborhood.*

Proof. To prove this, we let z_1 be any point other than z_0 in the given neighborhood. We then let ρ be the distance between z_1 and z_0. If C_ρ denotes the positively oriented circle $|z - z_0| = \rho$, centered at z_0 and passing through z_1 (Fig. 4-35), the Cauchy integral formula tells us that

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)dz}{z - z_0}; \tag{4.15.1}$$

and the parametric representation

$$z = z_0 + \rho e^{i\theta} \quad (0 \le \theta \le 2\pi)$$

for C_ρ enables us to write equation (4.15.1) as

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta. \tag{4.15.2}$$

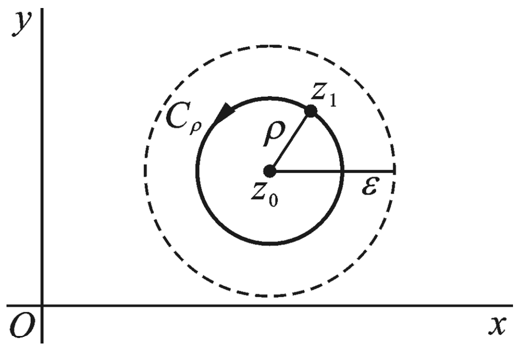

Fig. 4-35

From equation (4.15.2), we obtain the inequality

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|\, d\theta \le |f(z_0)|. \tag{4.15.3}$$

Hence

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta,$$

therefore,

$$\int_0^{2\pi} [|f(z_0)| - |f(z_0+\rho e^{i\theta})|]d\theta = 0.$$

The integrand in this last integral is continuous in the variable θ and larger than or equal to 0 on the entire interval $[0,2\pi]$. Then the integrand must be identically equal to zero. That is,

$$|f(z_0+\rho e^{i\theta})| = |f(z_0)| \quad (0 \le \theta \le 2\pi). \tag{4.15.6}$$

Especially, $|f(z_1)| = |f(z_0)|$. Since the point z_1 was arbitrary, we see that the equation $|f(z)| = |f(z_0)|$ is, in fact, satisfied by all points z in the neighborhood $|z - z_0| < \varepsilon$. But we know from Exercise $7(b)$, Sec. 2.14, that when the modulus of an analytic function is constant in a domain, the function itself is constant there. Thus $f(z) = f(z_0)$ for each point z in the neighborhood, and the proof is complete.

This lemma can be used to prove the following theorem, which is known as the m*aximum modulus principle* (MMP).

Theorem 4.15.1(Maximum Modulus Principle). *If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D. That is, there is no point z_0 in the domain such that $|f(z)| \le |f(z_0)|$ for all points z in D.*

Proof. We shall prove the theorem by assuming that $|f(z)|$ does have a maximum value at some point z_0 in D and then showing that $f(z)$ must be constant throughout D.

We draw a polygonal line L lying in D and extending from z_0 to any other point P in D. Also, d represents the shortest distance from points on L to the boundary of D. When D is the entire plane, d may have any positive value. Next, we observe that there is a finite sequence of points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n$ along L such that z_n coincides with the point P and $|z_k - z_{k-1}| < d$ $(k = 1, 2, \ldots, n)$. On forming a finite sequence of neighborhoods (Fig. 4-36)

$$N_0, N_1, N_2, \ldots, N_{n-1}, N_n,$$

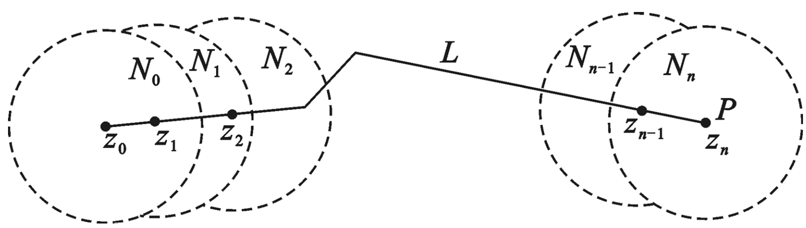

Fig. 4-36

where each N_k has center z_k and radius d, we see that f is analytic in each of these neighborhoods, which are all contained in D, and that the center of each neighborhood $N_k (k = 1, 2, \dots, n)$ lies in the neighborhood N_{k-1}.

Since $|f(z)|$ was assumed to have a maximum value in D at z_0, it also has a maximum value in N_0 at that point. Hence, according to Lemma 4.15.1, $f(z)$ has the constant value $f(z_0)$ on N_0. Particularly, $f(z_1) = f(z_0)$. This means that $|f(z)| \le |f(z_1)|$ for each point z in N_1; and Lemma 4.15.1 can be applied again, this time telling us that $f(z) = f(z_1) = f(z_0)$ whenever z is in N_1. Since z_2 is in N_1, then, $f(z_2) = f(z_0)$. Hence $|f(z)| \le |f(z_2)|$ when z is in N_2; and Lemma 4.15.1 is once again applicable, showing that $f(z) = f(z_2) = f(z_0)$ when z is in N_2. Continuing in this manner, we eventually reach the neighborhood N_n and arrive at the fact that $f(z_n) = f(z_0)$. Recalling that z_n coincides with the point P, which is any point other than z_0 in D, we may conclude that $f(z) = f(z_0)$ for every point z in D. Thus, $f(z)$ has been shown to be constant throughout D, the theorem is proved.

If a function f is analytic at each point in the interior of a closed bounded region R and is continuous throughout R, then the modulus $|f(z)|$ has a maximum value somewhere in R (Sec. 2.7). That is, there exists a point $z_0 \in R$ such that $|f(z)| \le |f(z_0)|$ for all point z in R. If f is a constant function, then $|f(z)| = |f(z_0)|$ for all z in R. If, however, $f(z)$ is not constant, then, according to the maximum modulus principle, $|f(z)| \ne |f(z_0)|$ for any point z in the interior of R. We thus arrive at an important corollary of the maximum modulus principle.

Corollary 4.15.1. *Suppose that a function f is continuous on a closed bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ on R occurs somewhere on the boundary of R and never in the interior of R.*

Example. Let R denote the rectangular region $0 \le x \le \pi, 0 \le y \le 1$. Corollary 4.15.1 tells us that the modulus of the entire function $f(z) = \sin z$ has a maximum value in R that occurs somewhere on the boundary, and not in the interior of R. This can be verified directly by writing

$$|f(z)|=\sqrt{\sin^2 x+\sinh^2 y}$$

and noting that, in R, the term $\sin^2 x$ is greatest when $x=\pi/2$ and that the increasing function $\sinh^2 y$ is greatest when $y=1$. Thus the maximum value of $|f(z)|$ in R occurs at the boundary point $z=(\pi/2,1)$ and at no other point in R (Fig. 4-37).

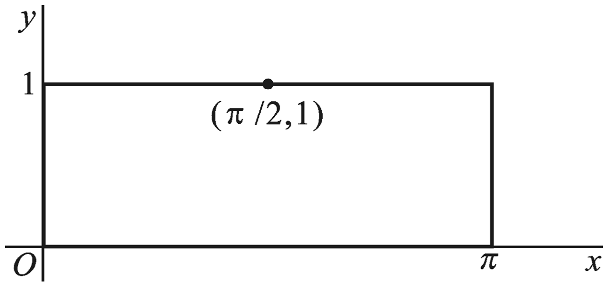

Fig. 4-37

When the function f in Corollary 4.15.1 is written as

$$f(z)=u(x,y)+iv(x,y),$$

the component function $u(x,y)$ also has a maximum value in R which is assumed on the boundary of R and never in the interior, where it is harmonic (Sec. 2.15). This is because that the composite function $g(z)=\exp[f(z)]$ is continuous in R and analytic and not constant in the interior. Consequently, its modulus $|g(z)|=\exp[u(x,y)]$, which is continuous in R, must assume its maximum value in R on the boundary. Because of the increasing nature of the exponential function, it follows that the maximum value of $u(x,y)$ also occurs on the boundary.

Properties of minimum values of $|f(z)|$ and $u(x,y)$ are treated in the exercises.