Chapter Ⅷ

Conformal Mappings

In this chapter, we introduce and develop the concept of a conformal mapping. The geometric interpretation of a function of a complex variable as a mapping, or transformation, was introduced in Secs. 2.2 and 2.3 (Chap. 2). We saw there how the nature of such a function can be displayed graphically, to some extent, by the manner in which it maps certain curves and regions. In this chapter, we shall see further examples of how various curves and regions are mapped by some elementary analytic functions.

§8.1. Conformal mappings

Definition 8.1.1. A transformation $w = f(z)$ is said to be *conformal* at a point z_0 if f is analytic there and $f'(z_0) \neq 0$.

Such a transformation is actually conformal at each point in a neighborhood of z_0. For f must be analytic in a neighborhood of z_0 (Sec. 2.13); and, since f is continuous at z_0 (Sec. 4.13), it follows from Theorem 2.7.2 in Sec. 2.7 that there is also a neighborhood of that throughout which $f'(z) \neq 0$.

Definition 8.1.2. A transformation $w = f(z)$ on a domain D is referred to as a *conformal transformation*, or *conformal mapping*, when it is conformal at each point in D.

Each of the elementary functions studied in Chap. 3 can be used to define a transformation that is conformal in some domain.

Example 1. The mapping $w = e^z$ is conformal throughout the entire z plane since $(e^z)' = e^z \neq 0$ for each z. Consider any two lines $x = c_1$ and $y = c_2$ in the z plane, the first directed upward and the second directed to the right. According to Sec. 2.3, their images under the mapping $w = e^z$ are a positively oriented circle centered at the origin and a ray from the origin, respectively. As illustrated in Fig. 2-5 (Sec. 2.3), the angle between the lines at their point of intersection is a right angle in the negative direction, and the same is true of the angle between the circle and the ray at the corresponding point in the w plane.

Example 2. Consider two smooth arcs which are level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ of the real and imaginary components, respectively, of a function

$$f(z) = u(x, y) + iv(x, y),$$

and suppose that they intersect at a point z_0 where f is analytic and $f'(z_0) \neq 0$. The transformation $w = f(z)$ is conformal at z_0 and maps the arcs into the lines $u = c_1$ and $v = c_2$, which are orthogonal at the point $w_0 = f(z_0)$ because of the Cauchy-Riemann equations. According to our theory, then, the arcs must be orthogonal at z_0. This has already been verified and illustrated in Exercises 7 through 11 of Sec. 2.15.

A mapping that preserves the magnitude of the angle between two smooth arcs but not necessarily the sense is called an *isogonal mapping*.

Example 3. The transformation $w = \overline{z}$, which is a reflection in the real axis, is isogonal but not conformal. If it is followed by a conformal transformation, the resulting transformation $w = f(\overline{z})$ is also isogonal but not conformal.

Suppose that f is not a constant function and is analytic at a point z_0. If, in addition, $f'(z_0)=0$, then z_0 is called a *critical point* of the transformation $w=f(z)$.

Example 4. The point $z=0$ is a critical point of the transformation

$$w=1+z^2,$$

which is a composition of the mappings

$$Z=z^2 \quad \text{and} \quad w=1+Z.$$

A ray $\theta=\alpha$ from the point $z=0$ is evidently mapped onto the ray from the point $w=1$ whose angle of inclination is 2α. Moreover, the angle between any two rays drawn from the critical point $z=0$ is doubled by the transformation.

More generally, it can be shown that if z_0 is a critical point of a transformation $w=f(z)$, there is an integer $m(m\geq 2)$ such that the angle between any two smooth arcs passing through z_0 is multiplied by m under that transformation. The integer m is the smallest positive integer such that $f^{(m)}(z_0)\neq 0$. Verification of these facts is left to the exercises.

Next, let us discuss the angle-preserving property of a conformal mapping. Let C be a smooth arc (Sec. 4.3), represented by the equation $z=z(t)$ $(a\leq t\leq b)$, and let $f(z)$ be a function defined at all points z on C. The equation $w=f[z(t)]$ $(a\leq t\leq b)$ is a parametric representation of the image Γ of C under the transformation $w=f(z)$.

Suppose that C passes through a point $z_0=z(t_0)$ $(a\leq t_0\leq b)$ at which f is conformal at z_0. According to the chain rule, if $w(t)=f[z(t)]$, then

$$w'(t_0)=f'[z(t_0)]z'(t_0); \tag{8.1.1}$$

and this means that (see Sec. 1.7)

$$\text{Arg}w'(t_0)=\text{Arg}\, f'[z(t_0)]z'(t_0)+\text{Arg}z'(t_0). \tag{8.1.2}$$

Statement (8.1.2) is useful in relating the directions of C and Γ at the points z_0 and $w_0=f(z_0)$, respectively.

To be specific, let ψ_0 denote a value of $\text{Arg}f'(z_0)$, and let θ_0 be the angle of inclination of a directed line tangent to C at z_0 (Fig. 8-1).

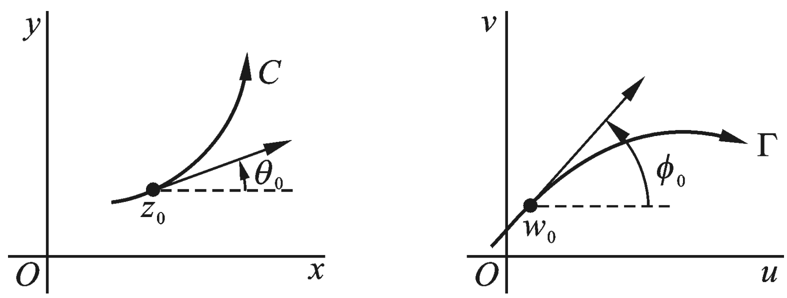

Fig. 8-1

According to Sec. 4.3, θ_0 is a value of the argument of $z'(t_0)$; and it follows from statement (8.1.2) that the quantity

$$\phi_0=\psi_0+\theta_0 \tag{8.1.3}$$

is a value of $\text{Arg}w'(t_0)$ and is, therefore, the angle of inclination of a directed line tangent to Γ at the point $w_0=f(z_0)$.

Now let C_1 and C_2 be two smooth arcs passing through z_0, and let θ_1 and θ_2 be angles of inclination of directed lines tangent to C_1 and C_2, respectively, at z_0. We know from (8.1.3) that the quantities

$$\phi_1 = \psi_0 + \theta_1 \quad \text{and} \quad \phi_2 = \psi_0 + \theta_2$$

are angles of inclination of directed lines tangent to the image curves Γ_1 and Γ_2, respectively, at the point $w_0 = f(z_0)$. Thus, $\phi_2 - \phi_1 = \theta_2 - \theta_1$; that is, the angle $\phi_2 - \phi_1$ from Γ_1 to Γ_2 is the same in magnitude and sense as the angle $\theta_2 - \theta_1$ from C_1 to C_2. Those angles are denoted by α in Fig. 8-2.

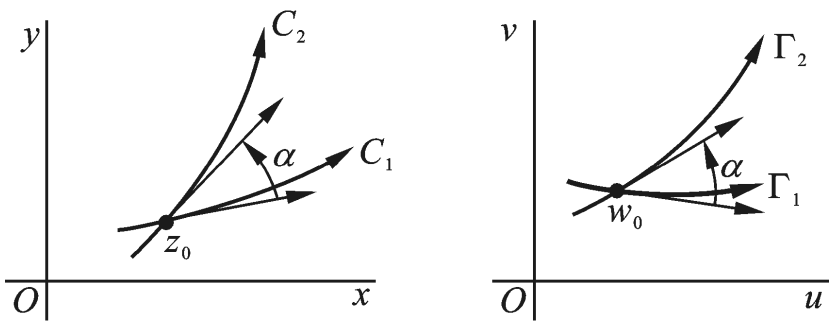

Fig. 8-2

Lastly, let us discuss the geormetrical meaning of the derivative $f'(z_0)$ of a conformal mapping f. To consider a transformation $w = f(z)$ that is conformal at a point z_0. From the definition of derivative, we know that

$$|f'(z_0)| = \left|\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}\right| = \lim_{z\to z_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}. \qquad (8.1.4)$$

Now $|z-z_0|$ is the length of a line segment joining z_0 and z, and $|f(z)-f(z_0)|$ is the length of the line segment joining the points $f(z_0)$ and $f(z)$ in the w plane. Evidently, then, if z is near the point z_0, the ratio

$$\frac{|f(z)-f(z_0)|}{|z-z_0|}$$

of the two lengths is approximately the number $|f'(z_0)|$. That is,

$$\frac{|f(z)-f(z_0)|}{|z-z_0|} \approx |f'(z_0)|, \text{ or } |f(z)-f(z_0)| \approx |f'(z_0)| \cdot |z-z_0|,$$

whenever $z \approx z_0$. Note that $|f'(z_0)|$ represents an expansion if it is greater than unity and a contraction if it is less than unity.

Although the angle of rotation $\arg f'(z_0)$ and the *scale factor* $|f'(z)|$ vary, in general, from point to point, it follows from the continuity of f' that their values are approximately $\arg f'(z_0)$ and $|f'(z_0)|$ at point z near z_0. Consequently, the image of a small region in a neighborhood of z_0 *conforms* to the original region in the sense that it has approximately the same shape. A large region may be transformed into a region that bears no resemblance to the original one.

Example 5. When $f(z) = z^2$, the transformation

$$w = f(z) = x^2 - y^2 + i2xy$$

is conformal at the point $z = 1 + i$, where the half lines $y = x\ (x \geq 0)$ and $x = 1\ (x \geq 0)$ intersect. We denote those half lines by C_1 and C_2, with positive sense upward, and observe that the angle from C_1 to C_2 is $\pi/4$ at their point of intersection (Fig. 8-3). Since the image of a point $z = (x, y)$ is a point in the w plane whose rectangular coordinates are

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy,$$

the half line C_1 is transformed into the curve Γ_1 with parametric representation

$$u = 0, \quad v = 2x^2 \qquad (0 \leq x < \infty). \tag{8.1.5}$$

Thus Γ_1 is the upper half $v \geq 0$ of the v axis. The half line C_2 is transformed into the curve Γ_2 represented by the equations

$$u = 1 - y^2, \quad v = 2y \qquad (0 \leq y < \infty). \tag{8.1.6}$$

Hence Γ_2 is the upper half of the parabola $v^2 = -4(u-1)$. Note that, in each case, the positive sense of the image curve is upward.

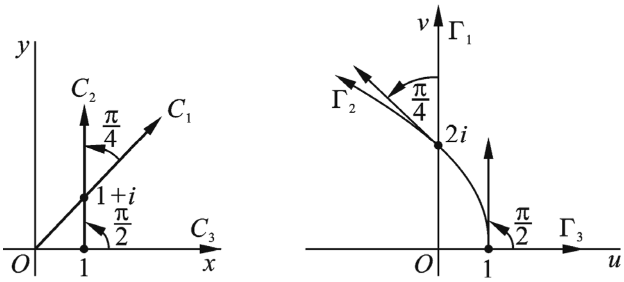

Fig. 8-3

If u and v are the vari tation (8.1.6) for the image curve Γ_2, then

$$\frac{dv}{du} = \frac{dv/dy}{du/dy} = \frac{2}{-2y} = -\frac{2}{v}.$$

In particular, $dv/du = -1$ when $v = 2$. Consequently, the angle from the image curve Γ_1 to the image curve Γ_2 at the point $w = f(1+i) = 2i$ is $\pi/4$, as required by the conformality of the mapping at $z = 1 + i$. As anticipated, the angle of rotation $\pi/4$ at the point $z = 1 + i$ is a value of

$$\text{Arg}[f'(1+i)] = \text{Arg}[2(1+i)] = \frac{\pi}{4} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The scale factor at that point is the number

$$|f'(1+i)| = |2(1+i)| = 2\sqrt{2}.$$

To illustrate how the angle of rotation and the scale factor can change from point to point, we note that they are 0 and 2, respectively, at the point $z = 1$ since $f'(1) = 2$. See Fig. 8-3, where the curves C_2 and Γ_2 are the one just discussed and where the nonnegative x axis C_3 is transformed into the nonnegative u axis Γ_3.