§7.3. Improper Integrals From Fourier Analysis

Residue theory can be useful in evaluating convergent improper integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin ax dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos ax dx, \tag{7.3.1}$$

where a denotes a positive constant. As in Sec. 7.1, we assume that $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. Also, $q(z)$ has no real zeros. Integrals of type (7.3.1) occur in the theory and application of the Fourier integral.

The method described in Sec. 7.1 and used in Sec. 7.2 cannot be applied directly here since (see Sec. 3.6)

$$|\sin az|^2 = \sin^2 ax + \sinh^2 ay$$

and

$$|\cos ax|^2 = \cos^2 ax + \sinh^2 ay .$$

More precisely, since

$$\sinh ay = \frac{e^{ay} - e^{-ay}}{2},$$

the moduli $|\sin az|$ and $|\cos az|$ increase like e^{ay} as y tends to infinity. The modification illustrated in the example below is suggested by the fact that

$$\int_{-R}^{R} f(x)\cos ax dx + i\int_{-R}^{R} f(x)\sin ax dx = \int_{-R}^{R} f(x)e^{iax} dx,$$

together with the fact that the modulus

$$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{-ay}e^{iax}| = e^{-ay}$$

is bounded in the upper half plane $y \ge 0$.

Example. Let us show that

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{2\pi}{e^3}. \tag{7.3.2}$$

Because the integrand is even, it is sufficient to show that the Cauchy principal value of the integral exists and to find that value.

We introduce the function

$$f(z) = \frac{1}{(z^2+1)^2} \tag{7.3.3}$$

and observe that the product $f(z)e^{i3z}$ is analytic everywhere on and above the real axis except at the point $z = i$. The singularity $z = i$ lies in the interior of the semicircular region whose boundary consists of the segment $-R \le x \le R$ of the real axis and the upper half C_R of the circle $|z| = R (R > 1)$ from $z = R$ to $z = -R$. Integration of $f(z)e^{i3z}$ around that boundary yields the equation

$$\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2} dx = 2\pi i B_1 - \int_{C_R} f(z)e^{i3z} dz, \tag{7.3.4}$$

where

$$B_1 = \operatorname*{Res}_{z=i}[f(z)e^{i3z}].$$

Since

$$f(z)e^{i3z} = \frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z) = \frac{e^{i3z}}{(z+i)^2},$$

the point $z = i$ is evidently a pole of order $m = 2$ of $f(z)e^{i3z}$; and

$$B_1 = \phi'(i) = \frac{1}{ie^3}.$$

By equating the real parts on each side of equation (7.3.4), then, we find that

$$\int_{-R}^{R} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{2\pi}{e^3} - \text{Re} \int_{C_R} f(z) e^{i3z} dz . \tag{7.3.5}$$

Finally, we observe that when z is a point on C_R,

$$| f(z) | \le M_R \quad \text{where} \quad M_R = \frac{1}{(R^2-1)^2}$$

and that $| e^{i3z} | = e^{-3y} \le 1$ for such a point. Consequently,

$$\left| \text{Re} \int_{C_R} f(z) e^{i3z} dz \right| \le \left| \int_{C_R} f(z) e^{i3z} dz \right| \le M_R \pi R \to 0, \tag{7.3.6}$$

as R tends to ∞ and because of inequalities (7.3.6), we need only let R tend to ∞ in equation (7.3.5) to arrive at the desired result (7.3.2).