Chapter Ⅱ

Analytic Functions

We now discuss complex-valued functions of a complex variable and develop a theory of differentiation for them. The main goal of the chapter is to introduce analytic functions, which play a central role in complex analysis.

§2.1. Functions of a Complex Variable

1.Definition of complex-valued functions of a complex variable

Let D be a set of complex numbers. A *function* f defined on D is a rule that assigns to each z in D a complex number w. The number w is called the *value* of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set D is called the *domain of definition* of f and set $f(D)$ is called the *range* of f. Since the variable z and the value $f(z)$ of a function f at z are all complex numbers, we call such a function a complex-valued functions of a complex variable.

2.Examples

Example 1. The function $w = 1/z$ is defined on the set $D = \{z \in \mathbf{C} : z \neq 0\}$.

For a complex-valued function f of complex variable $z = x + iy$ defined on D, put

$$u(x, y) = \operatorname{Re} f(x+iy), v(x, y) = \operatorname{Im} f(x+iy),$$

then we obtain two real-valued functions u and v defined on D so that $w = f(z)$ can be expressed in terms of a pair of real-valued functions of real variables x and y:

$$w = f(z) = u(x, y) + iv(x, y), \quad \forall z = x + iy \in D. \tag{2.1.1}$$

If the polar coordinates r and θ, instead of x and y, are used, then

$$w = f(z) = u(r, \theta) + iv(r, \theta), \quad \forall z = re^{i\theta} \in D, \tag{2.1.2}$$

where $u(r, \theta) = \operatorname{Re} f(re^{i\theta}), v(r, \theta) = \operatorname{Im} f(re^{i\theta})$.

Example 2. If $f(z) = z^2$, then

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + i2xy.$$

Hence

$$u(x,y) = x^2 - y^2 \text{ and } v(x,y) = 2xy.$$

When polar coordinates are used,

$$f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Consequently,

$$u(r,\theta) = r^2 \cos 2\theta \text{ and } v(r,\theta) = r^2 \sin 2\theta.$$

If, in either of equations (2.1.1) and (2.1.2), the function v always has value zero, then the value of f is always real. That is, f is a real-valued function of a complex variable.

Example 3. A real-valued function that is used to illustrate some important concepts later in this chapter is

$$w = f(z) = |z|^2 = x^2 + y^2 + i0.$$

If n is zero or a positive integer and if $a_0, a_1, a_2, \ldots, a_n$ are complex constants with $a_n \neq 0$, then the function

$$w = P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is called a *polynomial of degree* n. Note that the sum here has a finite number of terms and that the domain of definition is the entire z-plane. A quotient $P(z)/Q(z)$ of polynomials is called a *rational function* and are defined at each point z where $Q(z) \neq 0$. Polynomials and rational functions constitute elementary, but important, classes of functions of a complex variable.

A generalization of the concept of function, called a *multiple-valued function*, is a rule that assigns more than one value to a point z in the domain of definition. These *multiple-valued functions* occur in the theory of functions of a complex variable, just as they do in the case of real variables. When multiple-valued functions are studied, usually just one of the possible values assigned to each point is taken, in a systematic manner, and a single-valued function is constructed from the multi-valued function.

Example 4. Let z denote any nonzero complex number. We know from Sec. 1.8 that $z^{1/2}$ has the two values:

$$w = z^{1/2} = \pm\sqrt{r}\exp\left(i\frac{\theta}{2}\right),$$

where $r = |z|$ and $\theta(-\pi < \theta \le \pi)$ is the principal value of $\mathrm{Arg}z$. This formula gives a multiple-valued function, which is indeed two-valued. But, if we choose only the positive value of $\pm\sqrt{r}$ and write

$$w = f_+(z) = \sqrt{r}\exp\left(i\frac{\theta}{2}\right) \quad (r > 0, -\pi < \theta \le \pi), \tag{2.1.3}$$

then we obtain a single-valued function f_+ defined on the set of nonzero numbers in the z-plane. Since zero is the only square root of zero, we also write $f_+(0) = 0$. The function f_+ is then well defined on the entire plane. Similarly, we can get another single-valued function f_- as follows.

$$w = f_-(z) = -\sqrt{r}\exp\left(i\frac{\theta}{2}\right) \quad (r > 0, -\pi < \theta \le \pi). \tag{2.1.4}$$