§7.9. Rouche's Theorem

The main result in this section is known as *Rouche's theorem* and is a consequence of the argument principle. It can be useful in locating regions of the complex plane in which a given analytic function has zeros.

Theorem 7.9.1(Rouche). *Suppose that*

(i) f *and* g *are analytic inside and on a simple closed contour* C *;*

(ii) $|f(z)|>|g(z)|$ *at each point on* C.

Then f *and* $f+g$ *have the same number of zeros, counting multiplicities, inside* C.

Proof. We may assume that the orientation is positive. We begin with the observation that neither the function $f(z)$ nor the sum $f(z)+g(z)$ has a zero on C, since

$$|f(z)|>|g(z)|\geq 0$$

and

$$|f(z)+g(z)|\geq||f(z)|-|g(z)||>0$$

when z is on C.

If Z_f and Z_{f+g} denote the number of zeros, counting multiplicities, of $f(z)$ and $f(z)+g(z)$, respectively, inside C, we know from Theorem 7.8.1 that

$$Z_f=\frac{1}{2\pi}\Delta_C \arg f(z)$$

and

$$Z_{f+g}=\frac{1}{2\pi}\Delta_C \arg[f(z)+g(z)].$$

Consequently, since

$$\Delta_C \arg[f(z)+g(z)]=\Delta_C \arg\left\{f(z)\left[1+\frac{g(z)}{f(z)}\right]\right\}$$

$$=\Delta_C \arg f(z)+\Delta_C \arg\left[1+\frac{g(z)}{f(z)}\right],$$

it is clear that

$$Z_{f+g}=Z_f+\frac{1}{2\pi}\Delta_C \arg F(z), \tag{7.9.1}$$

where $F(z)=1+\dfrac{g(z)}{f(z)}$. But

$$|F(z)-1|=\frac{|g(z)|}{|f(z)|}<1;$$

so, under the map $w=F(z)$, the image of C lies in the open disk $|w-1|<1$. That image does not, then, enclose the origin $w=0$. Hence $\Delta_C \arg F(z)=0$ and, since equation (7.9.1) reduces to $Z_{f+g}=Z_f$, the proof is completed.

For an alternative proof of this theorem without using the argument principle, one can refer to Exercise 8, Sec. 7.9.

Example. In order to determine the number of roots of the equation

$$z^7-4z^3+z-1=0 \tag{7.9.2}$$

inside the circle $|z|=1$, write

$$f(z)=-4z^3 \quad\text{and}\quad g(z)=z^7+z-1.$$

Then observe that when $|z|=1$, we have

$$|f(z)|=4|z|^3=4$$

and

$$|g(z)| \le |z|^7 + |z| + 1 = 3.$$

The conditions in Rouche's theorem are thus satisfied. Consequently, since $f(z)$ has three zeros, counting multiplicities, inside the circle $|z| = 1$, so does $f(z) + g(z)$. That is, equation (7.9.2) has three roots there.