Chapter Ⅳ
Integrals

Integrals are extremely important in the study of functions of a complex variable. The theory of integration, to be developed in this chapter, is noted for its mathematical elegance. The theorems are generally concise and powerful, and most of the proofs are simple.

§4.1. Derivatives of Complex-Valued Functions of One Real Variable

In order to introduce integrals of f in a fairly simple way, we need to first consider derivative of a complex-valued function w of a *real* variable t. We write

$$w(t) = u(t) + iv(t), \tag{4.1.1}$$

where the functions u and v are *real-valued* functions of a real variable t.

Definition 4.1.1. If the derivatives $u'(t)$ and $v'(t)$ exists at t, then we say that the function (4.1.1) is *differentiable,* or *derivable,* at t and its *derivative* $w'(t)$, or $d[w(t)]/dt$, at t is defined as

$$w'(t) = u'(t) + iv'(t). \tag{4.1.2}$$

From definition (4.1.2), it follows that if a function $w(t) = u(t) + iv(t)$ is differentiable at t, then for every complex constant $z_0 = x_0 + iy_0$, the function $z_0 w$ is differentiable at t, and

$$\begin{aligned}\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u(t) + iv(t))]' \\ &= [(x_0 u(t) - y_0 v(t)) + i(y_0 u(t) + x_0 v(t))]' \\ &= (x_0 u(t) - y_0 v(t))' + i(y_0 u(t) + x_0 v(t))' \quad . \\ &= (x_0 u'(t) - y_0 v(t))' + i(y_0 u'(t) + x_0 v'(t)) \\ &= (x_0 + iy_0)(u' + iv') \\ &= z_0 w'(t).\end{aligned}$$

So, $z_0 w$ is differentiable at t and

$$\frac{d}{dt}[z_0 w(t)] = z_0 w'(t). \tag{4.1.3}$$

Another expected rule that we shall often use is

$$\frac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t}, \tag{4.1.4}$$

where $z_0 = x_0 + iy_0$. To verify this, we write

$$e^{z_0 t} = e^{x_0 t} e^{iy_0 t} = e^{x_0 t} \cos y_0 t + ie^{x_0 t} \sin y_0 t$$

and refer to definition (4.1.2) to see that

$$\frac{d}{dt} e^{z_0 t} = (e^{x_0 t} \cos y_0 t)' + i(e^{x_0 t} \sin y_0 t)'.$$

Familiar rules from calculus and some simple algebra then lead us to the expression

$$\frac{d}{dt} e^{z_0 t} = (x_0 + iy_0)(e^{x_0 t} \cos y_0 t + ie^{x_0 t} \sin y_0 t).$$

This gives that

$$\frac{d}{dt} e^{z_0 t} = z_0 e^{x_0 t} e^{iy_0 t} = z_0 e^{tz_0}.$$

and eqution (4.1.4) is established.

Various other rules learned in calculus, such as the ones for differentiating sums and products, apply just as they do for real-valued functions of t. As was the case with property (4.1.3) and formula (4.1.4), verifications may be based on corresponding rules in calculus. It should be pointed out, however, that not every rule for derivatives in calculus carries over to functions of type (4.1.1). The following example illustrates this.

Example. Suppose that the function w given by (4.1.1), is continuous on an interval $[a,b]$; that is, its component functions u and v are continuous there. Even if $w'(t)$ exist when $a < t < b$, the mean value theorem for derivatives no longer applies. To be precise, it is not necessarily true that there is a number c in the interval (a,b) such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}.$$

To see this, consider the function $w(t) = e^{it}$ on the interval $[0,2\pi]$. When that function is used, we have $|w'(t)| = |ie^{it}| = 1$; and this means that the derivative w' is never zero, while

$$w(2\pi) - w(0) = 0.$$

This shows that the desired number c doed not exist for this function.