§4.14. Liouville's Theorem

This section is devoted to some important theorems that follow from Theorem 4.13.1.

Theorem 4.14.1. *Suppose that* f *is analytic inside and on a positively oriented circle* C_R, *centered at* z_0 *and with radius* R (Fig. 4-34). *If* M_R *denotes the maximum value of* $|f(z)|$ *on* C_R, *then*

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M_R}{R^n} \quad (n=1,2,\ldots).\ (\textit{Cauchy's inequality}) \qquad (4.14.1)$$

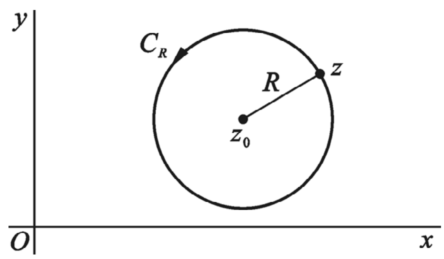

Fig. 4-34

Proof. From Theorem 4.13.1, we see that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R}\frac{f(z)dz}{(z-z_0)^{n+1}}\,(n=1,2,\ldots).$$

Applying inequality (4.6.1) yields that

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\cdot\frac{M_R}{R^{n+1}}2\pi R \quad (n=1,2,\ldots).$$

This inequality is, of course, the same as inequality (4.14.1). This completes the proof.

The result above can be used to show that no entire function except a constant is bounded in the complex plane. Our first theorem here, which is known as Linouville's theorem, states this result in a somewhat different way.

Theorem 4.14.2(Liouville). *If* f *is entire and bounded in the complex plane, then* $f(z)$ *is constant throughout the plane.*

Proof. Let $|f(z)| \le M < \infty (\forall z \in \mathbf{C})$. Since f is entire, Cauchy's inequality (4.14.1) with $n=1$ holds for any choices of z_0 and R:

$$\left|f'(z_0)\right| \le \frac{M_R}{R} \le \frac{M}{R} \to 0 (R\to\infty). \qquad (4.14.2)$$

Hence $f'(z_0)=0$. Since the choice of z_0 was arbitrary, this means that $f'(z)=0$ for all points z in the complex plane. According to the Theorem 2.13.1, f is a constant function. The proof is completed.

The following theorem, known as the fundamental theorem of algebra, follows readily from Liouville's theorem.

Theorem 4.14.3(Fundamental Theorem of Algebra). *Any polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad (a_n \ne 0)$$

of degree $n(n\ge 1)$ *has at least one zero. That is, there is a point* z_0 *such that* $P(z_0)=0$.

Proof. The proof here is by contradiction. Suppose that $P(z)$ is not zero for any value of z. Then the reciprocal $f(z)=\dfrac{1}{P(z)}$ is clearly entire. To show that f is bounded, we first write

$$w(z)=\frac{a_0}{z^n}+\frac{a_1}{z^{n-1}}+\frac{a_2}{z^{n-2}}+\cdots+\frac{a_{n-1}}{z}, \tag{4.14.4}$$

so that $P(z)=(a_n+w(z))z^n$. Since $\lim_{z\to\infty}|w(z)|=0$, we can choose a sufficiently large positive numebr R such that $|w(z)|<\frac{|a_n|}{2}$ when $|z|\geq R$. Consequently, when $|z|\geq R$,

$$|a_n+w(z)|\geq|a_n|-|w(z)|>\frac{|a_n|}{2},$$

and so

$$|P(z)|=|a_n+w(z)||z^n|>\frac{|a_n|}{2}|z|^n\geq\frac{|a_n|}{2}R^n \tag{4.14.5}$$

whenever $|z|\geq R$. Thus, we have

$$|f(z)|=\frac{1}{|P(z)|}\leq\frac{2}{|a_n|R^n} \quad \text{whenever } |z|>R.$$

Since f is continuous on the closed disk $|z|\leq R$, it is bounded there too. Hence f is bounded in the entire plane. It follows from Liouville's theorem that $f(z)$, and consequently $P(z)$, is constant. But $P(z)$ is indeed not constant, and we have reached a contradiction. The proof is completed now.

The fundamental theorem of algebra tells us that my polynomial $P(z)$ of degree $n(n\geq 1)$ can be expressed as a product of linear factors:

$$P(z)=c(z-z_1)(z-z_2)\cdots(z-z_n), \tag{4.14.6}$$

where c and $z_k(k=1,2,\dots,n)$ are complex constants. More precisely, the theorem ensures that $P(z)$ has a zero z_1. Then, according to Exercise 7, Sec. 4.15, $P(z)=(z-z_1)Q_1(z)$, where $Q_1(z)$ is a polynomial of degree $n-2$. Continuing in this way, we arrive at expression (4.14.6). Some of the constants z_k in expression (4.14.6) may, of course, appear more than once, and it is clear that $P(z)$ can have no more than n distinct zeros.