§6.2. Cauchy's Residue Theorem

If, except for a *finite* number of singular points, a function f is analytic inside a simple closed path C, those singular points must be isolated (Sec. 6.1). The following theorem, which is known as *Cauchy's residue theorem*, is a precise statement of the fact that if f is also analytic on C and if C is positively oriented, then the value of the integral of f around C is $2\pi i$ times the sum of the residues of f at the singular points inside C.

Theorem 6.2.1(Cauchy). *Let C be a simple closed path, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points $z_k(k=1,2,\ldots,n)$ inside C, then*

$$\int_C f(z)dz = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z). \qquad (6.2.1)$$

Proof. To prove the theorem, let the point $z_k(k=1,2,\ldots,n)$ be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common (Fig. 6-5). The circles C_k, together with the simple closed path C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain. Hence, according to the Theorem 4.11.2, Sec. 4.11, we have

$$\int_C f(z)dz - \sum_{k=1}^{n}\int_{C_k} f(z)dz = 0.$$

This reduces to equation (6.2.1) because (Sec. 6.1)

$$\int_{C_k} f(z)dz = 2\pi i\operatorname*{Res}_{z=z_k} f(z) \quad (k=1,2,\ldots,n),$$

and the proof is complete.

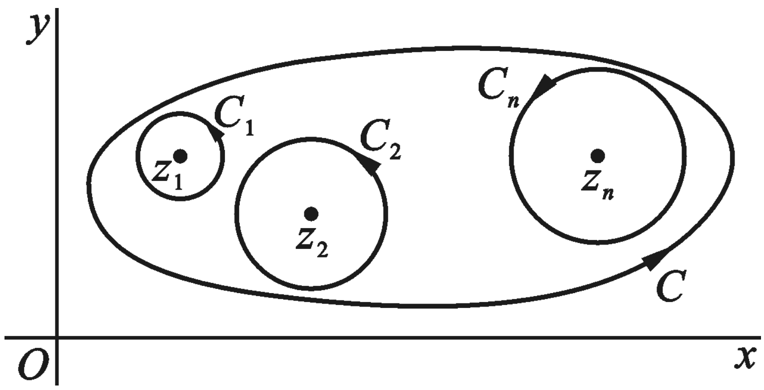

Fig. 6-5

Example 1. Use the Cauchy's residue theorem to evaluate the integral

$$\int_C \frac{5z-2}{z(z-1)}dz$$

when C is the circle $|z|=2$, described counterclockwise.

Solution. The integrand has the two isolated singularities $z=0$ and $z=1$, both of which are interior to C. We can find the residues B_1 at $z=0$ and B_2 at $z=1$ with the aid of the Maclaurin series

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \quad (|z|<1).$$

We observe first that when $0<|z|<1$ (Fig. 6-6),

$$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z} \cdot \frac{-1}{1-z} = \left(5 - \frac{2}{z}\right)(-1 - z - z^2 - \cdots);$$

and, by identifying the coefficient of $1/z$ in the product on the right here, we find that $B_1 = 2$. Also, since

$$\frac{5z-2}{z(z-1)} = \frac{5(z-1)+3}{z-1} \cdot \frac{1}{1+(z-1)} = \left(5 + \frac{3}{z-1}\right)[1 - (z-1) + (z-1)^2 - \cdots],$$

when $0 < |z-1| < 1$, it is clear that $B_2 = 3$. Thus

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i(B_1 + B_2) = 10\pi i.$$

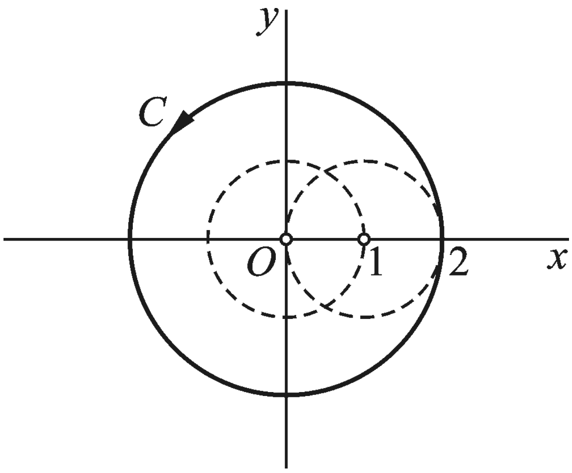

Fig. 6-6