§4.13. Derivatives of Analytic Functions

The Cauchy integral formula (Sec. 4.12) can be used to prove that if a function is analytic at a point, then its derivatives of all orders of the function exist at that point and are themselves analytic there. To prove this, we start with a lemma that extends the Cauchy integral formula so as to apply to derivatives of the first and second order.

Lemma 4.13.1. *Suppose that a function f is analytic everywhere inside and on a simple closed path C, taken in the positive sense, then*

$$f'(z)=\frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2},\quad \forall z\in \text{ins}(C). \tag{4.13.1}$$

Note that expressions (4.13.1) can be obtained formally, or without rigorous verification, by differentiating with respect to z under the integral sign in the Cauchy integral formula

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(s)ds}{s-z}, \tag{4.13.2}$$

where z is interior to C and s denotes points on C.

Proof. To verify the first of expressions (4.13.1), we let $z\in \text{ins}(C)$ and $d=\text{dist}(z,C)$ denote the smallest distance from z to points on C and use formula (4.13.2) to write

$$\begin{aligned}\frac{f(z+\Delta z)-f(z)}{\Delta z}&=\frac{1}{2\pi i}\int_C\left(\frac{1}{s-z-\Delta z}-\frac{1}{s-z}\right)\frac{f(s)}{\Delta z}ds\\&=\frac{1}{2\pi i}\int_C\frac{f(s)ds}{(s-z-\Delta z)(s-z)},\end{aligned}$$

where $0<|\Delta z|<d$ (see Fig. 4-32). Evidently, then,

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_C\frac{f(s)}{(s-z)^2}ds=\frac{1}{2\pi i}\int_C\frac{\Delta z f(s)}{(s-z-\Delta z)(s-z)^2}ds. \tag{4.13.3}$$

Next, we let $M=\max\{|f(s)|: s\in C\}$. Observe that $\forall s\in C$ we have $|s-z|\ge d$ and $|\Delta z|<d$, and therefore

$$|s-z-\Delta z|=|(s-z)-\Delta z|\ge |s-z|-|\Delta z|\ge d-|\Delta z|>0.$$

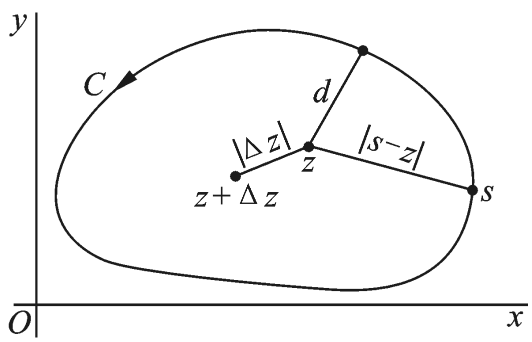

Fig. 4-32

Thus $\forall s\in C$, we get

$$\left|\frac{\Delta z f(s)}{(s-z-\Delta z)(s-z)^2}\right|\le \frac{|\Delta z|M}{(d-\Delta z)d^2}.$$

This shows that

$$\left|\int_C \frac{\Delta z f(s)ds}{(s-z-\Delta z)(s-z)^2}\right| \le \frac{|\Delta z| M}{(d-|\Delta z|)d^2} L,$$

where L is the length of C. Upon letting Δz tend to zero, we find from this inequality that the right-hand side of equation (4.13.3) also tends to zero. Consequently,

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2};$$

and the desired expression for $f'(z)$ is established. The proof is complete.

Theorem 4.13.1. *Suppose that a function* f *is analytic everywhere inside and on a simple closed path* C*, taken in the positive sense. Then for* $n = 1,2,\dots$,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}}, \ \forall z \in \text{ins}(D). \tag{4.13.4}$$

Proof. We will complete the proof by induction. By Lemma 4.13.1, we know that the formula (4.13.4) is valid for $n = 1$. Assume that (4.13.4) is valid for $n-1$, that is,

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^n}, \quad \forall z \in \text{ins}(C). \tag{4.13.5}$$

Let $z \in \text{ins}(C)$ be fixed, and

$$d = \inf_{s\in C} |s-z|, D = \max_{s\in C} |s-z|, M = \max_{s\in C} |f(s)|. \tag{4.13.6}$$

Let $0 < |h| < d$. Then $z + h \in \text{ins}(C)$ and so by (4.13.5), we have

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} - \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}}$$

$$= \frac{n!}{2\pi i}\int_C \left[\frac{1}{nh}\frac{(s-z)^n - (s-z-h)^n}{(s-z)^n(s-z-h)^n} - \frac{1}{(s-z)^{n+1}}\right] f(s)ds.$$

To estimate the integrand of the last integral above , we put

$$\varphi(s,h) = \frac{1}{nh}\frac{(s-z)^n - (s-z-h)^n}{(s-z)^n(s-z-h)^n} - \frac{1}{(s-z)^{n+1}},$$

then

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} - \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}} = \frac{n!}{2\pi i}\int_C \varphi(s,h) f(s)ds. \tag{4.13.7}$$

We write

$$(s-z-h)^n = (s-z)^n - n(s-z)^{n-1}h + \cdots + h^n = (s-z)^n - n(s-z)^{n-1}h + h\cdot g(s,h),$$

where $g(s,h) = \sum_{k=2}^{n} C_n^k (-1)^k (s-z)^{n-k} h^{k-1}$. It is clear that $\forall s \in C$,

$$|g(s,h)| \le \sum_{k=2}^{n} C_n^k D^{n-k} |h|^{k-1} \le |h| \cdot \sum_{k=2}^{n} C_n^k D^{n-k} d^{k-1} = K \cdot |h|, \tag{4.13.8}$$

where $K = \sum_{k=2}^{n} C_n^k D^{n-k} d^{k-1}$ is a positive constant. Using (4.13.8), we obtain that $\forall s \in C$,

$$|\varphi(s,h)f(s)| = \left|\frac{1}{nh}\frac{n(s-z)^{n-1}h - h\cdot g(s,h)}{(s-z)^n(s-z-h)^n} - \frac{1}{(s-z)^{n+1}}\right| |f(s)|$$

$$= \frac{|(s-z)^n - \frac{1}{n}(s-z)g(s,h) - (s-z-h)^n|}{|(s-z)^{n+1}(s-z-h)^n|} |f(s)|$$

$$= \frac{|n(s-z)^{n-1}h - h\cdot g(s,h) - \frac{1}{n}(s-z)g(s,h)|}{|(s-z)^{n+1}(s-z-h)^n|} |f(s)|$$

$$\le \frac{nD^{n-1} + K|h| + DK}{d^{n+1}(d-|h|)^n}\cdot |h| M.$$

It follows from (4.13.7) that

$$\left|\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} - \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}}\right| \le \frac{n!}{2\pi}\frac{nD^{n-1} + K|h| + DK}{d^{n+1}(d-|h|)^n}\cdot |h| ML,$$

where L denotes the length of C. Hence,

$$\lim_{h\to 0}\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^{n+1}}.$$

This shows that the formula (4.13.4) is valid for n and completes the proof.

Theorem 4.13.2. *If a function f is analytic at a point z_0, then its derivatives of all orders of f exist and are analytic there.*

Proof. Since the function f is analytic at z_0, there must be a neghborhood $N(z_0,\varepsilon)$ of z_0 throughout which f is analytic (see Sec. 2.13). Let C_0 be the circle centered at z_0 and with radius $\varepsilon/2$, then f is analytic inside and on C_0 (Fig. 4-33). According to Theorem 4.13.1 above, derivatives $f^{(n)}(z)$ exist and

$$f^{(n)}(z) = \frac{n!}{\pi i}\int_{C_0}\frac{f(s)ds}{(s-z)^{n+1}}$$

for all points z interior to C_0 and all positive integers n. The existence of $f''(z)$ throughout the neighborhood $|z-z_0| < \varepsilon/2$ means that f' is analytic at z_0. One can apply the same argument to the analytic function f' to conclude that its derivative f'' is analytic, etc. Theorem 4.13.1 is now established.

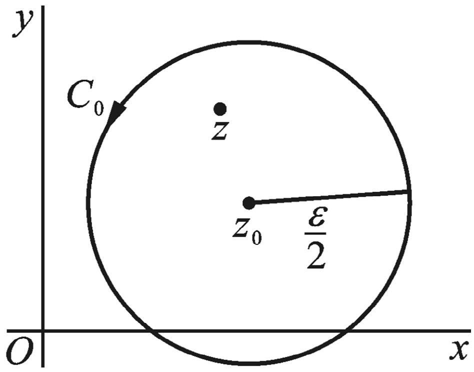

Fig. 4-33

As a consequence, when a function

$$f(z) = u(x,y) + iv(x,y)$$

is analytic at a point $z=(x,y)$, the differentiability of f' ensures the continuity of f' there (Sec. 2.8). Thus, since

$$f'(z)=u_x+iv_x=v_y-iu_y,$$

we may conclude that the first-order partial derivatives of u and v are continuous at that point. Furthermore, since f'' is analytic and continuous at z and since

$$f''(z)=u_{xx}+iv_{xx}=v_{yx}-iu_{yx},$$

etc., we arrive at a corollary that was anticipated in Sec. 2.15, where harmonic functions were introduced.

Corollary 4.13.1. *If a function $f=u+iv$ is analytic at a point $z=(x,y)$, then the component functions u and v have continuous partial derivatives of all orders at that point.*

Note that, with the agreement that

$$f^{(0)}(z)=f(z) \text{ and } 0!=1,$$

expression (4.13.4) is also valid when $n=0$, in which case it becomes the Cauchy integral formula (4.12.1).

When written in the form

$$\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}=\frac{2\pi i}{n!}f^{(n)}(z_0)(n=1,2,\ldots), \tag{4.13.9}$$

expression (4.13.4) can be used in evaluating certain integrals when f is analytic inside and on a simple closed path C, taken in the positive sense, and z_0 is any point interior to C. It has already been illustrated in Sec. 4.12 when $n=0$.

Example 1. If C is the positively oriented unit circle $|z|=1$ and $f(z)=\exp(2z)$, then

$$\int_C \frac{\exp(2z)dz}{z^4}=\int_C \frac{f(z)dz}{(z-0)^{3+1}}=\frac{2\pi i}{3!}f'''(0)=\frac{8\pi i}{3}.$$

Example 2. Let z_0 be any point interior to a positively oriented simple closed path C. When $f(z)=1$, expression (4.13.9) shows that

$$\int_C \frac{dz}{z-z_0}=2\pi i \text{ and } \int_C \frac{dz}{(z-z_0)^{n+1}}=0(n=1,2,\ldots),$$

Example 3. From (4.13.9), we get

$$\int_{|z|=1} \frac{\sin z dz}{z^{n+1}}=\frac{2\pi i}{n!}\sin^{(n)}(0)=\frac{2\pi i}{n!}\sin\frac{n\pi}{2}(n=1,2,\ldots).$$

We conclude this section with a theorem due to E. Morera (1856-1909). The proof here depends on the fact that the derivative of an analytic function is itself analytic, as stated in Theorem 4.13.2.

Theorem 4.13.3(Morera). *Let f be continuous in a domain D. If*

$$\int_C f(z)dz=0 \tag{4.13.10}$$

for every simple closed path C lying in D, then f is analytic throughout D.

Proof. At fisrt, we observe that when its hypothesis is satisfied, the Theorem 4.7.1 ensures that f has a primitive function in D; that is, there exists an analytic function F such that

$$F'(z)=f(z) \quad \forall z\in D.$$

Since f is the derivative of F, it follows from Theorem 4.13.2 above that f is analytic in D.

Corollary 4.13.2. *Let f be continuous on a simply connected domain D, then f is analytic in D if and only if*

$$\int_C f(z)dz=0$$

for every simple closed path C lying in D.