§5.10. Multiplication and Division of Power Series

Suppose that each of the power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n (z-z_0)^n \tag{5.10.1}$$

converges within some circle $|z-z_0| = R$. Their sums $f(z)$ and $g(z)$, respectively, are then analytic functions in the disk $|z-z_0| < R$ (Sec. 5.8), and the product of those sums has a Taylor series expansion which is valid there:

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \qquad (|z-z_0| < R). \tag{5.10.2}$$

According to Theorem 5.9.1 in Sec. 5.9, the series (5.10.1) are themselves Taylor series. Hence the first three coefficients in series (5.10.2) are given by the equations

$$c_0 = f(z_0)g(z_0) = a_0 b_0,$$

$$c_1 = \frac{f(z_0)g'(z_0) + f'(z_0)g(z_0)}{1!} = a_0 b_1 + a_1 b_0,$$

and

$$c_2 = \frac{f(z_0)g''(z_0) + 2f'(z_0)g'(z_0) + f''(z_0)g(z_0)}{2!} = a_0 b_2 + a_1 b_1 + a_2 b_0.$$

The general expression for any coefficient c_n is easily obtained by referring to Leibniz's rule (Exercise 6)

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z), \tag{5.10.3}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $(k = 0, 1, 2, \ldots, n)$, for the *n*th derivative of the product of two differentiable functions. As usual, $f^{(0)}(z) = f(z)$ and $0! = 1$. Evidently,

$$c_n = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k};$$

and so expansion (5.10.2) can be written

$$f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0)(z-z_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(z-z_0)^2 + \cdots$$

$$+\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(z-z_0)^n + \cdots \quad (|z-z_0| < R). \tag{5.10.4}$$

Series (5.10.4) is the same as the series obtained by formally multiplying the two series (5.10.1) term by term and collecting the resulting terms in like powers of $z - z_0$; it is called the *Cauchy product* of the two given series.

Example1. The function $e^z/(1+z)$ has only a singular point at $z=-1$, and so its Maclaurin series representation is valid in the open disk $|z|<1$. The first three nonzero terms are easily found by writing

$$\frac{e^z}{1+z} = e^z \frac{1}{1-(-z)}$$

$$= \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots\right)(1 - z + z^2 - z^3 + \cdots)$$

and multiplying these two series term by term. To be precise, we may multiply each term in the first series by 1, then each term in that series by $-z$, etc. The following systematic approach is suggested, where like powers of z are assembled vertically so that their coefficients can be readily added:

$$\begin{array}{l} 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots \\ \quad - z - z^2 - \frac{1}{2}z^3 - \frac{1}{6}z^4 - \cdots \\ \qquad\quad z^2 + z^3 + \frac{1}{2}z^4 + \frac{1}{6}z^5 - \cdots \\ \qquad\qquad - z^3 - z^4 - \frac{1}{2}z^5 - \frac{1}{6}z^6 - \cdots \\ \qquad\qquad\quad \vdots \end{array}$$

The desired result is

$$\frac{e^z}{1+z} = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \cdots \quad (|z|<1). \tag{5.10.5}$$

Continuing to let $f(z)$ and $g(z)$ denote the sums of series (5.10.1), suppose that $g(z) \neq 0$ when $|z-z_0|<R$. Since the quotient $f(z)/g(z)$ is analytic throughout the disk $|z-z_0|<R$, it has a Taylor series representation

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z-z_0)^n \quad (|z-z_0|<R), \tag{5.10.6}$$

where the coefficients d_n can be found by differentiating $f(z)/g(z)$ successively and evaluating the derivatives at $z = z_0$. The results are the same as those found by formally carrying out the division of the first of series (5.10.1) by the second.

Example 2. As pointed out in Sec. 3.7, the zeros of the entire function $\sinh z$ are the numbers $z = n\pi i \ (n = 0, 1, 2, \ldots)$. So the quotient

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3}\left(\frac{1}{1 + z^2/3! + z^4/5! + \cdots}\right), \tag{5.10.7}$$

has a Laurent series representation in $0 < |z| < \pi$. The denominator of the fraction in parentheses on the right-hand side of (5.10.7) is a power series that converges to $(\sinh z)/z$ when $z \neq 0$ and to 1 when $z = 0$. A power series representation of the fraction can be found by dividing the series into unity as follows:

$$\begin{array}{r|l}
 & 1 - \frac{1}{3!}z^2 + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots \\
\hline
1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots & 1 \\
 & 1 + \frac{1}{3!}z^2 + \frac{1}{5!}z^4 + \cdots \\
\cline{2-2}
 & -\frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \cdots \\
 & -\frac{1}{3!}z^2 - \frac{1}{(3!)^2}z^4 - \cdots \\
\cline{2-2}
 & \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots \\
 & \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots \\
\cline{2-2}
 & \vdots
\end{array}$$

Thus,

$$\frac{1}{1 + z^2/3! + z^4/5! + \cdots} = 1 - \frac{1}{3!}z^2 + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^4 + \cdots.$$

Hence

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\cdot\frac{1}{z} + \frac{7}{360}z + \cdots \qquad (0 < |z| < \pi). \tag{5.10.9}$$