## §8.5. The Transformation $w = 1/z$

The equation

$$w = \frac{1}{z} \tag{8.5.1}$$

establishes a one to one correspondence between the nonzero points of the $z$ and the $w$ planes. Since $z\bar{z} = |z|^2$, the mapping can be described by means of the successive transformations

$$Z = \frac{1}{|z|^2} z, \quad w = \bar{Z}. \tag{8.5.2}$$

The first of these transformations is an *inversion* with respect to the unit circle $|z| = 1$. That is, the image of a nonzero point $z$ is the point $Z$ with the properties

$$|Z| = \frac{1}{|z|} \text{ and } \arg Z = \arg z.$$

Thus the points exterior to the circle $|z| = 1$ are mapped onto the nonzero points interior to it (Fig. 8-5), and conversely. Any point on the circle is mapped onto itself. The second of transformations (8.5.2) is simply a *reflection* in the real axis.

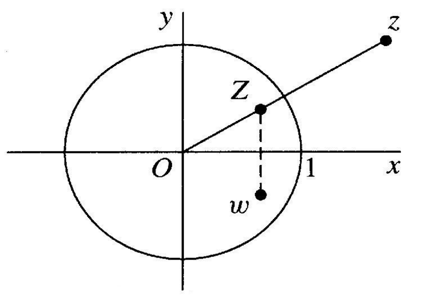


**Fig. 8-5**

If we write transformation (8.5.1) as

$$T(z) = \frac{1}{z} \quad (z \neq 0), \tag{8.5.3}$$

we can define $T$ at the origin and at the point at infinity so as to be continuous on the *extended* complex plane. To do this, we need only refer to Sec.2.6 to see that

$$\lim_{z \to 0} T(z) = \infty \text{ since } \lim_{z \to 0} \frac{1}{T(z)} = 0 \tag{8.5.4}$$

and

$$\lim_{z \to \infty} T(z) = 0 \text{ since } \lim_{z \to 0} T\left(\frac{1}{z}\right) = 0. \tag{8.5.5}$$

In order to make $T$ continuous on the extended plane, we define

$$T(0) = \infty, \quad T(\infty) = 0. \tag{8.5.6}$$

More precisely, equations (8.5.6), together with the first of limits (8.5.4) and (8.5.5), show that

$$\lim_{z \to z_0} T(z) = T(z_0) \tag{8.5.7}$$

for every point $z_0$ in the extended plane, including $z_0 = 0$ and $z_0 = \infty$.