§7.2. Examples

We turn now to an illustration of the method in Sec. 7.1 for evaluating improper integrals.

Example. In order to evaluate the integral $\int_0^\infty \frac{x^2}{x^6+1}dx$, we start with the observation that the function $f(z)=\frac{z^2}{z^6+1}$ has isolated singularities at the zeros of z^6+1, which are the sixth roots of -1, and is analytic everywhere else. The method in Sec. 1.8 for finding roots of complex numbers reveals that the sixth roots of -1 are $c_k=\exp\left[i\left(\frac{\pi}{6}+\frac{2k\pi}{6}\right)\right]$ $(k=0,1,2,\ldots,5)$, and it is clear that none of them lies on the real axis. The first three roots, $c_0=e^{i\pi/6}, c_1=i$ and $c_2=e^{i5\pi/6}$
lie in the upper half plane (Fig. 7-2) and the other three lie in the lower one.

Fig. 7-2

When $R>1$, the points $c_k\,(k=0,1,2)$ lie in the interior of the semicircular region bounded by the segment $z=x(-R\le x\le R)$ of the real axis and the upper half C_R of the circle $|z|=R$ from $z=R$ to $z=-R$. Integrating $f(z)$ counterclockwise around the boundary of this semicircular region, we see that

$$\int_{-R}^{R} f(x)dx+\int_{C_R} f(z)dz=2\pi i(B_0+B_1+B_2)\,, \tag{7.2.1}$$

where B_k is the residue of $f(z)$ at $c_k\,(k=0,1,2)$.

With the aid of Theorem 6.8.2 in Sec.6.8, we find that the points c_k are simple poles of f and that

$$B_k=\operatorname*{Res}_{z=c_k}\frac{z^2}{z^6+1}=\frac{c_k^2}{6c_k^5}=\frac{1}{6c_k^3}\quad (k=0,1,2)\,.$$

Thus

$$2\pi i(B_0+B_1+B_2)=2\pi i\left(\frac{1}{6i}-\frac{1}{6i}+\frac{1}{6i}\right)=\frac{\pi}{3}\,;$$

and equation (7.2.1) can be put in the form

$$\int_{-R}^{R} f(x)dx=\frac{\pi}{3}-\int_{C_R} f(z)dz\,, \tag{7.2.2}$$

which is valid for all values of R greater that 1.

Next, we show that the value of the integral on the right in equation (7.2.2) tends to 0 as R tends to ∞. To do this, we observe that when $|z|=R$,

$$|z^2|=|z|^2=R^2$$

and

$$|z^6+1| \geq ||z|^6-1| = R^6-1.$$

So, if z is any point on C_R,

$$|f(z)| = \frac{|z^2|}{|z^6+1|} \leq M_R \quad \text{where} \quad M_R = \frac{R^2}{R^6-1};$$

and this means that

$$\left|\int_{C_R} f(z)dz\right| \leq M_R \pi R = \frac{\pi R^3}{R^6-1} \to 0, \tag{7.2.3}$$

as R tends to ∞. Thus, $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$. It now follows from equation (7.2.2) that

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{x^6+1}dx = \frac{\pi}{3},$$

that is

$$P.V.\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}dx = \frac{\pi}{3}.$$

Since the integrand here is even, we know from equations (7.1.6) in Sec.7.1 and Theorem 7.1.2 that

$$\int_{0}^{\infty} \frac{x^2}{x^6+1}dx = \frac{\pi}{6}. \tag{7.2.4}$$