§5.3. Examples

When it is known that f is analytic everywhere inside a circle centered at z_0, the convergence of its Taylor series about z_0 to $f(z)$ for each point z within that circle is ensured; no test for the convergence of the series is required. In fact, according to Taylor's theorem, the series converges to $f(z)$ within the circle about z_0 whose radius is the distance from z_0 to the nearest point z_1 where f fails to be analytic. In Sec. 5.8, we shall find that this is actually the largest circle centered at z_0 such that the series converges to $f(z)$ for all z interior to it.

Also, in Sec. 5.9, we shall see that if there are constants $a_n (n = 0,1,2\ldots)$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all points z interior to some circle centered at z_0, then the power series here must be the Taylor series for f about z_0, regardless of how those constants arise. This observation often allows us to find the coefficients a_n in Taylor series in more efficient ways that by appealing directly to the formula $a_n = f^{(n)}(z_0)/n!$ in Taylor's theorem.

In the following expansions, we use the formula in Taylor's theorem to find the Maclaurin series expansions of some fairly simple functions, and we emphasize the use of those expansions in finding other representations. In our example, we shall freely use expected properties of convergent series, such as those verified in Exercises 3 and 4, Sec. 5.1.

Example 1. For $f(z) = e^z$, we have $f^{(n)}(z) = e^z$ and $f^{(n)}(0) = 1$. Thus,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (|z| < \infty). \tag{5.3.1}$$

The entire function $z^2 e^{3z}$ also has a Maclaurin series expansion. The simplest way to obtain it is to replace z by $3z$ on each side of equation (5.3.1) and then multiply through the resulting equation by z^2:

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} \quad (|z| < \infty).$$

Finally, if we replace n by $n-2$ here, we have

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \quad (|z| < \infty).$$

Example 2. One can use expansion (5.3.1) and the definition (Sec. 3.6)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

to find the Maclaurin series for the entire function $f(z) = \sin z$. To give the details, we refer to expansion (5.3.1) and write

$$\sin z = \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] = \frac{1}{2i}\sum_{n=0}^{\infty}[1-(-1)^n]\frac{i^n z^n}{n!} \quad (|z| < \infty).$$

But $1-(-1)^n = 0$ when n is even, and so we can replace n by $2n+1$ in this last series:

$$\sin z = \frac{1}{2i}\sum_{n=0}^{\infty}[1-(-1)^{2n+1}]\frac{i^{2n+1} z^{2n+1}}{(2n+1)!} \quad (|z| < \infty).$$

Inasmuch as

$$1-(-1)^{2n+1} = 2, \quad i^{2n+1} = (i^2)^n i = (-1)^n i,$$

this reduces to

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty). \tag{5.3.2}$$

Term by term differentiation will be justified in Sec. 5.8 using that procedure here, we differentiate each side of equation (5.3.2) and write

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d}{dz} z^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)!} z^{2n} .$$

Thus,

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad (|z| < \infty). \tag{5.3.3}$$

Example 3. Because $\sinh z = -i \sin(iz)$ (Sec. 3.7), we need only replace z by iz on each side of equation (5.3.2) and multiply through the result by $-i$ to see that

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty). \tag{5.3.4}$$

Likewise, since $\cosh z = \cos(iz)$, it follows from expansion (5.3.3) that

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad (|z| < \infty). \tag{5.3.5}$$

Observe that the Taylor series for $\cosh z$ about the point $z_0 = -2\pi i$ can be obtained by replacing the variable z by $z + 2\pi i$ on each side of equation (5.3.5) and then recalling that $\cosh(z + 2\pi i) = \cosh z$ for all z:

$$\cosh z = \sum_{n=0}^{\infty} \frac{(z + 2\pi i)^{2n}}{(2n)!} \quad (|z| < \infty).$$

Example 4. Another Maclaurin series representation is

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1). \tag{5.3.6}$$

The derivatives of the function $f(z) = 1/(1-z)$, which fails to be analytic at $z = 1$, are

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \quad (n = 0,1,2\ldots);$$

and, in particular, $f^{(n)}(0) = n!$. Note that expansion (5.3.6) gives us the sum of an infinite *geometric* series, where z is the *common ratio* of adjacent terms:

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z} \quad (|z| < 1).$$

This is, of course, the summation formula that was found in another way in the example is Sec. 5.1.

If we substitute $-z$ for z in equation (5.3.6) and its condition of validity, and note that $|z| < 1$ when $|-z| < 1$, we see that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad (|z| < 1).$$

If, on the other hand, we replace the variable z in equation (5.3.6) by $1-z$, we have the Taylor series representation

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad (|z-1| < 1).$$

This condition of validity follows from the one associated with expansion (5.3.6) since $|1-z| < 1$ is the same as $|z-1| < 1$.

Example 5. For our final example, let us expand the function

$$f(z)=\frac{1+2z^2}{z^3+z^5}=\frac{1}{z^3}\cdot\frac{2(1+z^2)-1}{1+z^2}=\frac{1}{z^3}\left(2-\frac{1}{1+z^2}\right)$$

into a series involving powers of z. We cannot find a Maclaurin series for $f(z)$ since it is not analytic at $z=0$. But we do know from expansion (5.3.6) that

$$\frac{1}{1+z^2}=1-z^2+z^4-z^6+z^8-\cdots \quad (|z|<1).$$

Hence, when $0<|z|<1$,

$$f(z)=\frac{1}{z^3}(2-1+z^2-z^4+z^6-z^8+\cdots)$$
$$=\frac{1}{z^3}+\frac{1}{z}-z+z^3-z^5+\cdots.$$

We call such terms as $1/z^3$ and $1/z$ negative powers of z since they can be written z^{-3} and z^{-1}, respectively. The theory of expansions involving negative powers of $z-z_0$ will be discussed in the next section.