§1.2. Basic Algebraic Properties

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

1. Commutative law

$$z_1+z_2= z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \tag{1.2.1}$$

2. Associative law

3. Distributive law

$$z(z_1 + z_2) = zz_1 + zz_2 , \tag{1.2.3}$$

$$nz = \overbrace{z + z + \cdots + z}^{n} \text{ and } z^n = \overbrace{zz\cdots z}^{n} .$$

4. Identities

The *additive identity* $0 = (0,0)$ and the *multiplicative identity* $1 = (1,0)$ for real numbers carry over to the entire complex number system. That is,

$$z + 0 = z \text{ and } z \cdot 1 = z \tag{1.2.4}$$

for every complex number z . Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 9).

5. Additive inverse

For each complex number $z = (x, y)$, there is an *additive inverse*

$$- z = (-x,-y) , \tag{1.2.5}$$

6. Subtraction

$$z_1 - z_2 =z_1+(-z_2), \quad \forall z_1, z_2 \in \mathbf{C} . \tag{1.2.6}$$

So if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 - z_2 = (x_1 - x_2, y_1 - y_2) = (x_1 - x_2) + i(y_1 - y_2) . \tag{1.2.7}$$

7. Multiplicative inverse

For $z = (x, y) = x + iy \neq 0$, there is a number z^{-1} such that $zz^{-1} = 1$, called the *multiplicative inverse* of z . It is easy to find that the multiplicative inverse of $z = (x, y) = x + iy$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} \quad (z \neq 0) . \tag{1.2.8}$$

From the discussion above, we conclude that the set $\mathbf{C}$ of all complex numbers becomes a *field*, called the *field of complex numbers*, or the *complex number field.*