§8.6. Mappings by $1/z$

When a point $w = u + iv$ is the image of a nonzero point $z = x + iy$ under the transformation $w = 1/z$, writing $w = \overline{z}/|z|^2$ reveals that

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}. \tag{8.6.1}$$

Also, since $z = 1/w = \overline{w}/|w|^2$,

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}. \tag{8.6.2}$$

Theorem 8.6.1 *The mapping $w = 1/z$ transforms circles and lines into circles and lines.*

Proof. When A, B, C, and D are all real numbers satisfying the condition

$$B^2 + C^2 > 4AD,$$

the equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \tag{8.6.3}$$

represents an arbitrary circle or line, where $A \neq 0$ for a circle and $A = 0$ for a line. The need for the condition $B^2 + C^2 > 4AD$ when $A \neq 0$ is evident if, by the method of completing the squares, we rewrite equation (8.6.3) as

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2.$$

When $A = 0$, the condition becomes $B^2 + C^2 > 0$, which means that B and C are not both zero. To prove the theorem, we observe that if x and y satisfy equation (8.6.3), we can use relations (8.6.2) to substitute for those variables. After some simplifications, we find that u and v satisfy the equation (see also Exercise 11 below)

$$D(u^2 + v^2) + Bu - Cv + A = 0, \tag{8.6.4}$$

which also represents a circle or line. This completes the proof.

It is now clear from equations (8.6.3) and (8.6.4) that

(i) a circle $(A \neq 0)$ not passing through the origin $(D \neq 0)$ in the z plane is transformed into a circle not passing through the origin in the w plane;

(ii) a circle $(A \neq 0)$ through the origin $(D = 0)$ in the z plane is transformed into a line that does not pass through the origin in the w plane;

(iii) a line $(A = 0)$ not passing through the origin $(D \neq 0)$ in the z plane is transformed into a circle through the origin in the w plane;

(iv) a line $(A = 0)$ through the origin $(D = 0)$ in the z plane is transformed into a line through the origin in the w plane.

Example 1. According to (8.6.3) and (8.6.4), a vertical line $x = c_1 (c_1 \neq 0)$ is transformed by $w = 1/z$ into the circle $-c_1(u^2 + v^2) + u = 0$, i.e.,

$$\left(u - \frac{1}{2c_1}\right)^2 + v^2 = \left(\frac{1}{2c_1}\right)^2, \tag{8.6.5}$$

which is centered on the u axis and tangent to the v axis. The image of a typical point (c_1, y) on the line is, by equations (8.6.1),

$$(u, v) = \left(\frac{c_1}{c_1^2 + y^2}, \frac{-y}{c_1^2 + y^2}\right).$$

If $c_1 > 0$, the circle (8.6.5) is evidently to the right of the v axis. As the point (c_1, y) moves up the entire line, its image traverses the circle once in the clockwise direction, the point at infinity in the extended z plane corresponding to the origin in the w plane. For if $y < 0$, then $v > 0$; and, as y increases through negative value to 0, we see that u increases from 0 to $1/c_1$. Then, as y increases through positive values, v is negative and u decreases to 0.

If, on the other hand, $c_1 < 0$, the circle lies to the left of the v axis. As the point (c_1, y) moves upward, its image still makes one cycle, but in the counterclockwise direction. See Fig. 8-6, where the cases $c_1 = 1/3$ and $c_1 = -1/2$ are illustrated.

Example 2. A horizontal line $y = c_2 (c_2 \neq 0)$ is mapped by $w = 1/z$ onto the circle

$$u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2, \tag{8.6.6}$$

which is centered on the v axis and tangent to the u axis. Two special cases are shown in Fig.8-6, where the corresponding orientations of the lines and circles are also indicated.

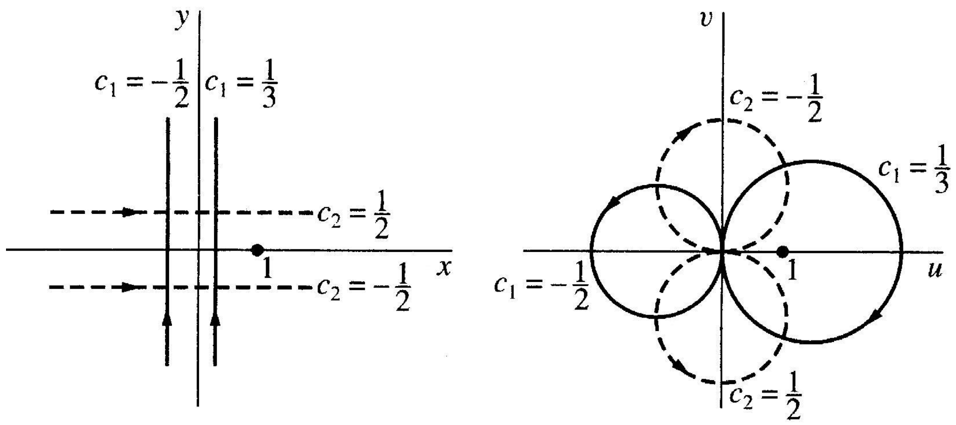

Fig. 8-6

Example 3. When $w = 1/z$, the half plane $x \geq c_1$ $(c_1 > 0)$ is mapped onto the disk

$$\left(u - \frac{1}{2c_1}\right)^2 + v^2 \leq \left(\frac{1}{2c_1}\right)^2. \tag{8.6.7}$$

For, according to Example 1, any line $x = c$ $(c \geq c_1)$ is transformed into the circle

$$\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2. \tag{8.6.8}$$

Furthermore, as c increases through all values greater than c_1, the lines $x = c$ move to the right and the image circles (8.6.8) shrink in size. (See Fig. 8-7.) Since the lines $x = c$ pass through all points in the half plane $x \geq c_1$ and the circles (8.6.8) pass through all points in the disk (8.6.7), the mapping is established.

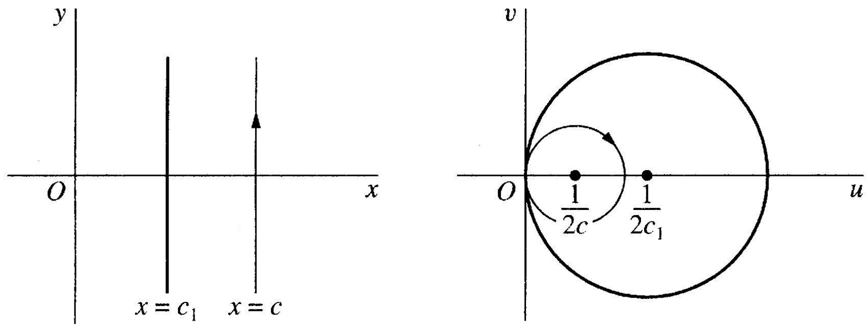

Fig. 8-7