§5.4. Laurent Series

If a function f fails to be analytic at point z_0, we cannot apply Taylor's theorem at that point. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $z-z_0$, see Example 5, Exercises 10,11, Sec. 5.3. We now present the theory of such representations, and we begin with Laurent's theorem.

Theorem 5.4.1(Laurent). *Suppose that a function f is analytic in an annular domain $R_1<|z-z_0|<R_2$ where $0\le R_1<R_2\le\infty$ and let C denote any positively oriented simple closed path around z_0 and lying in that domain* (Fig. 5-3). *Then $f(z)$ has the series representation*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}\quad (R_1<|z-z_0|<R_2), \tag{5.4.1}$$

where

$$a_n=\frac{1}{2\pi i}\int_C\frac{f(s)ds}{(s-z_0)^{n+1}}\quad (n=0,1,2\ldots) \tag{5.4.2}$$

and

$$b_n=\frac{1}{2\pi i}\int_C\frac{f(s)ds}{(s-z_0)^{-n+1}}\quad (n=0,1,2\ldots). \tag{5.4.3}$$

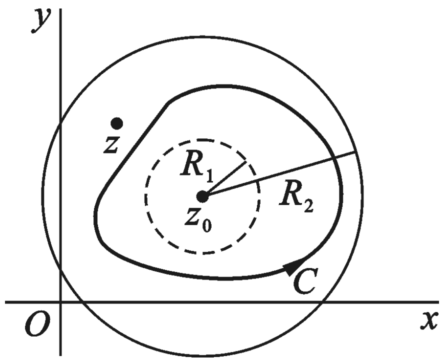

Fig. 5-3

Proof. We shall prove Laurent's theorem first when $z_0=0$, in which case the *annulus* is centered at the origin. The verification of the theorem when z_0 is arbitrary will follow readily.

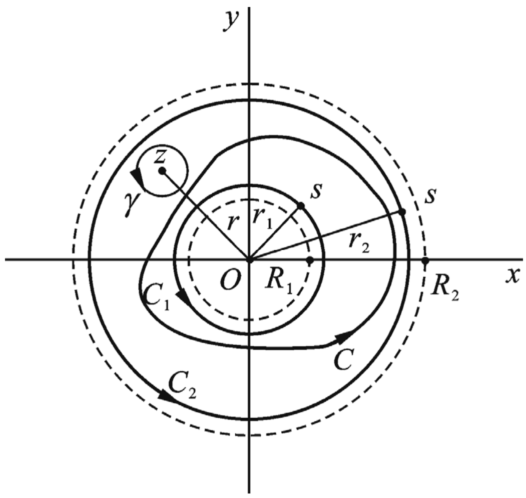

Fig. 5-4

Let z be any point such $R_1 < |z| < R_2$. Take r_1, r_2 such that the closed annular region $r_1 \le |z| \le r_2$ is contained in the domain $R_1 < |z| < R_2$ and whose interior contains both the point z and the path C (Fig. 5-4). We let C_1 and C_2 denote the circles $|z| = r_1$ and $|z| = r_2$, respectively, and we assign those two circles a positive orientation. Observe that f is analytic on C_1 and C_2, as well as in the annular domain between them.

Next, we construct a positively oriented circle γ with center at z and small enough to be completely contained in the interior of the annular region $r_1 \le |z| \le r_2$, as shown in Fig. 5-4. It then follows from the ECIT-2 (Theorem 4.11.2) that

$$\int_{C_2} \frac{f(s)ds}{s-z} - \int_{C_1} \frac{f(s)ds}{s-z} - \int_{\gamma} \frac{f(s)ds}{s-z} = 0.$$

But, according to the Cauchy integral formula, the value of third integral here is $2\pi i f(z)$. Hence

$$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{s-z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)ds}{z-s}. \tag{5.4.4}$$

Now the factor $1/(s-z)$ in the first of these integrals has the expansion

$$\frac{1}{s-z} = \sum_{k=0}^{n-1} \frac{1}{s^{k+1}} z^k + z^n \frac{1}{(s-z)s^n}. \tag{5.4.5}$$

As for the factor $1/(z-s)$ in the second integral, an interchange of s and z in equation (5.4.5) reveals that

$$\frac{1}{z-s} = \sum_{k=0}^{n-1} \frac{1}{s^{-k}} \cdot \frac{1}{z^{k+1}} + \frac{1}{z^n} \cdot \frac{s^n}{z-s}.$$

If we replace the index of summation k here by $k-1$, this expansion takes the form

$$\frac{1}{z-s} = \sum_{k=1}^{n} \frac{1}{s^{-k+1}} \cdot \frac{1}{z^k} + \frac{1}{z^n} \cdot \frac{s^n}{z-s}, \tag{5.4.6}$$

which is to be used in what follows.

Multiplying through equations (5.4.5) and (5.4.6) by $f(s)/(2\pi i)$ and then integrating each side of the resulting equations with respect to s around C_2 and C_1, respectively, we find from expression (5.4.4) that

$$f(z) = \sum_{k=0}^{n-1} a_k z^k + \rho_n(z) + \sum_{k=1}^{n} \frac{b_k}{z^k} + \sigma_n(z), \tag{5.4.7}$$

where the numbers $a_k (k = 0,1,2,\ldots,n-1)$ and $b_k (k = 0,1,2,\ldots,n)$ are given by the equations

$$a_k = \frac{1}{2\pi i}\int_{C_2} \frac{f(s)ds}{s^{k+1}}, \quad b_k = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)ds}{s^{-k+1}} \tag{5.4.8}$$

and where

$$\rho_n(z) = \frac{z^n}{2\pi i}\int_{C_2} \frac{f(s)ds}{(s-z)s^n}, \qquad \sigma_n(z) = \frac{1}{2\pi i z^n}\int_{C_1} \frac{s^n f(s)ds}{z-s}.$$

As n tends to ∞, expression (5.4.7) evidently takes the proper form of a Laurent series in the domain $R_1 < |z| < R_2$, provided that

$$\lim_{n\to\infty} \rho_n(z) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sigma_n(z) = 0. \tag{5.4.10}$$

These limits are readily established by a method already used in the proof of Taylor's theorem in Sec. 5.2. We write $|z| = r$, so that $r_1 < r < r_2$, and let M denote the maximum value of $|f(s)|$ on C_1 and C_2. We also note that if s is a point on C_2, then $|s-z| \ge r_2 - r$, and if

s is on C_1, $|z-s| \geq r - r_1$. This enables us to write

$$|\rho_n(z)| \leq \frac{Mr_2}{r_2 - r}\left(\frac{r}{r_2}\right)^n \text{ and } |\sigma_n(z)| \leq \frac{Mr_1}{r - r_1}\left(\frac{r_1}{r}\right)^n .$$

Since $(r/r_2) < 1$ and $(r_1/r) < 1$, it is now clear that (5.4.11) is valid.

Finally, we need only recall Corollary 4.11.2 in Sec. 4.11 to see that the paths used in integrals (5.4.8) may be replaced by the path C. Thus, (5.4.8) becomes (5.4.2) and (5.4.3) when $z_0 = 0$.

To extend the proof to the general case, we write $g(z) = f(z+z_0)$. Since $f(z)$ is analytic in the annulus $R_1 < |z - z_0| < R_2$, the function $f(z+z_0)$ is analytic when $R_1 < |(z+z_0) - z_0| < R_2$. That is, g is analytic in the annuls $R_1 < |z| < R_2$, which is centered at the origin. Now, the simple closed path C in the theorem has some parametric representation $z = z(t)(a \leq t \leq b)$, where

$$R_1 < |z(t) - z_0| < R_2 \tag{5.4.10}$$

for all t in the interval $a \leq t \leq b$. Hence, if Γ denotes the path

$$z = z(t) - z_0 \quad (a \leq t \leq b), \tag{5.4.11}$$

then Γ is not only a simple closed path but, in view of inequalities (5.4.10), it lies in the domain $R_1 < |z| < R_2$. Consequently, $g(z)$ has a Laurent series representation

$$f(z+z_0) = g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \quad (R_1 < |z| < R_2), \tag{5.4.12}$$

where

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)dz}{z^{n+1}} \quad (n = 0,1,2,\ldots), \tag{5.4.13}$$

$$b_n = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)dz}{z^{-n+1}} \quad (n = 1,2,\ldots). \tag{5.4.14}$$

Replacing z by $z - z_0$ (5.4.12) yields (5.4.1). Lastly, expression (5.4.13) for the coefficients a_n is, moreover, the same as expression (5.4.2) since

$$\int_\Gamma \frac{g(z)dz}{z^{n+1}} = \int_a^b \frac{f[z(t)]z'(t)}{[z(t)-z_0]^{n+1}}dt = \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} .$$

Similarly, (5.4.14) is the same as (5.4.3). This completes the proof.

Lastly, some remarks are given below.

Remark 1. Expansion (5.4.1) is often written

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad (R_1 < |z - z_0| < R_2) \tag{5.4.15}$$

where

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2 \ldots). \tag{5.4.16}$$

The series in (5.4.1) or (5.4.15) is called the *Laurent series* of f.

Remark 2. Observe that the integrand in expression (5.4.3) can be written $f(z)(z-z_0)^{n-1}$, which is analytic throughout the disk $|z - z_0| < R_2$ since f is. Hence, all of the coefficients b_n are zero; and, because (Sec. 4.13)

$$\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0,1,2\ldots),$$

expansion (5.4.1) reduces to a Taylor expansion of f about z_0.

Remark 3. If f is not analytic at z_0 but is analytic in the deleted disk $0 < |z - z_0| < R_2$, the radius R_1 can be chosen arbitrarily small. Representation (5.4.1) is then valid in the *punctured disk* $0 < |z - z_0| < R_2$. Similarly, if f is analytic at each point in the plane exterior to the circle $|z - z_0| = R_1$, the condition of validity is $R_1 < |z - z_0| < \infty$. Observe that if f is analytic everywhere in the plane except at z_0, expansion (5.4.1) is valid for each point z satisfying $0 < |z - z_0| < \infty$.