§2.11. Necessary and Sufficient Conditions for Differentiability

Satisfaction of the Cauchy-Riemann equations at a point $z_0 = (x_0, y_0)$ is not sufficient to ensure the existence of the derivative of a function $f(z)$ at that point. But, with certain differentiability conditions, we have the following useful theorem.

1. Theorem

Theorem 2.11.1. *Suppose that the function* $f(z) = u(x, y) + iv(x, y)$ *is defined on some neighborhood* $N(z_0, \varepsilon)$ *of a point* $z_0 = x_0 + iy_0$, *then* f *is differentiable at* z_0 *if and only if the functions* u *and* v *are differentiable at* (x_0, y_0) *and satisfy the Cauchy-Riemann equations* $u_x = v_y$, $u_y = -v_x$ *at* (x_0, y_0).

Corollary 2.11.1. *Let the function* $f(z) = u(x, y) + iv(x, y)$ *be defined on an open set* D *of the plane and suppose that the functions* u *and* v *have continuous partial derivatives and satisfy the Cauchy-Riemann equations at every point of* D, *then* f *is differentiable in* D *and*

$$f'(z) = u_x(x, y) + iv_x(x, y) = -i[u_y(x, y) + iv_y(x, y)], \quad \forall z = x + iy \in D.$$

By Theorem 2.11.1, we get the following corollary.

Corollary 2.11.2. *A function* $f(z) = u(x, y) + iv(x, y)$ *is differentiable in an open set* D *if and only if the functions* u *and* v *are differentiable and satisfy the Cauchy-Riemann equations in* D.

2. Examples

Example 1. Consider the exponential function

$$f(z) = e^z = e^x e^{iy} \quad (z = x + iy),$$

some of whose mapping properties were discussed in Sec.2.3. In view of Euler's formula (Sec. 1.6), this function can, of course, be written

$$f(z) = e^x \cos y + ie^x \sin y,$$

where y is to be taken in radians when $\cos y$ and $\sin y$ are evaluated. Then the real and imaginary parts of the exponential function are

$$u(x, y) = e^x \cos y \text{ and } v(x, y) = e^x \sin y,$$

respectively. Since $u_x = v_y$ and $u_y = -v_x$ everywhere and since these derivatives are everywhere continuous, the conditions in Corollary 2.11.1 are satisfied at all points in the complex plane. Thus, $f'(z)$ exists everywhere, and

$$f'(z) = u_x(x, y) + iv_x(x, y) = e^x \cos y + ie^x \sin y = f(z), \forall z = x + iy \in \mathbf{C}.$$

Note that $f'(z) = f(z), \forall z \in \mathbf{C}$.

Example 2. It also follows from Corollary 2.11.2 that the function $f(z) = |z|^2$, whose components are

$$u(x, y) = x^2 + y^2 \text{ and } v(x, y) = 0,$$

has a derivative at $z = 0$.

In fact, $f'(0) = 0 + i0 = 0$. We saw in Example 2, Sec. 2.10, that this function cannot have a derivative at any nonzero point since the Cauchy-Riemann equations are not satisfied at such points.