§2.2. Mappings

1. Definition of a mapping

One can, however, display some information about the function by indicating pairs of corresponding points $z=(x,y)$ and $w=(u,v)$. To do this, it is generally simpler to draw the z-plane and w-plane, separately. See the following figure.

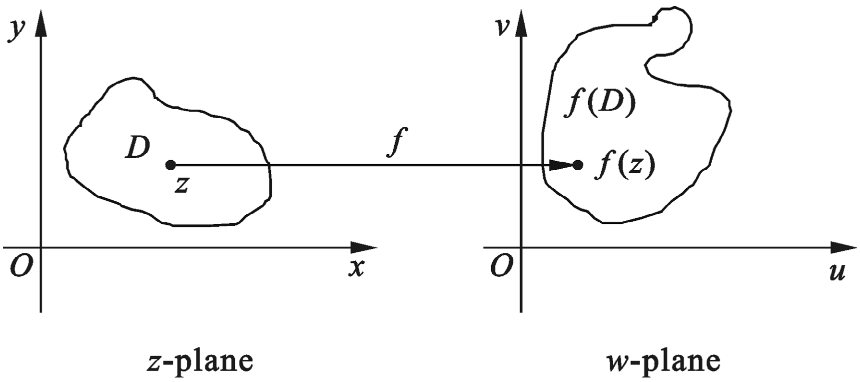

Fig. 2-1

When a function f is thought of in this way, it is often referred to as a *mapping*, or *transformation*. The image of a point z in the domain of definition D is the point $w=f(z)$, and the set of images of all points in a set T contained in D is called the *image of* T and denoted by $f(T)$. Thus,

$$f(T)=\{f(z): z\in T\}.$$

The image $f(D)$ of the entire domain of definition D is just the range of f and denoted by $\mathrm{ran}(f)$. The *inverse image* of a point w is the set of all points z in the domain of definition of f that have w as their image. The inverse image of a point may contain just one point, many points, or none at all. The last case occurs, of course, when w is not in the range of f. For a subset B of the plane, the set of all inverse images of the points in B is called the inverse image of B, denoted by $f^{-1}(B)$. Thus, $f^{-1}(B)=\{z\in D: f(z)\in B\}$.

2. Examples

Example 1. According to Example 2 in Sec. 2.1, the mapping $w=z^2$ can be thought of as the transformation

$$u=x^2-y^2, \quad v=2xy \tag{2.2.1}$$

from the xy-plane to the uv-plane. This form of the mapping is especially useful in finding the images of certain hyperbolas.

It is easy to show, for instance, that each branch of a hyperbola

$$x^2-y^2=c_1 \quad (c_1>0) \tag{2.2.2}$$

is mapped in a one to one manner onto the vertical line $u=c_1$. We start by noting from the first of equations (2.2.1) that $u=c_1$ when (x,y) is a point lying on either branch. When, in particular, it lies on the right-hand branch, the second of equations (2.2.1) tells us that $v=2y\sqrt{y^2+c_1}$. Thus the image of the right-hand branch can be expressed parametrically as

$$u=c_1, \quad v=2y\sqrt{y^2+c_1} \quad (-\infty<y<\infty);$$

and it is evident that the image of a point (x,y) on that branch moves upward along the entire line as (x, y) traces out the branch in the upward direction (Fig. 2-2). Likewise, since the pair of equations

$$u = c_1, \quad v = -2y\sqrt{y^2 + c_1} \quad (-\infty < y < \infty)$$

furnishes a parametric representation for the image of the left-hand branch of the hyperbola, the image of a point going downward along the entire left-hand branch is seen to move up the entire line $u = c_1$.

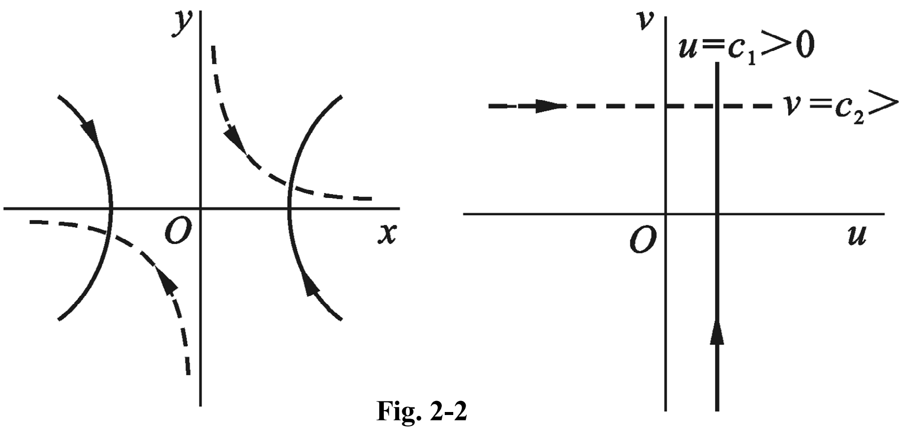

Fig. 2-2

On the other hand, each branch of a hyperbola

$$2xy = c_2 \quad (c_2 > 0) \tag{2.2.3}$$

is transformed into the line $v = c_2$, as indicated in Fig. 2-1.

We shall now use Example 1 to find the image of a certain region.

Example 2. The domain $D = \{(x, y): x > 0, y > 0, xy < 1\}$ consists of all points lying on the upper branches of hyperbolas from the family $2xy = c$, where $0 < c < 2$ (Fig. 18). We know from Example 1 that as a point travels downward along the entirety of one of these branches, its image under the transformation $w = z^2$ moves to the right along the entire line $v = c$. Since, for all value of c between 0 and 2, the branches fill out the domain

$$\{(x, y): x > 0, y > 0, xy < 1\},$$

that domain is mapped onto the horizontal strip $\{(u, v): 0 < v < 2\}$.

In view of equations (2.2.1), the image of a point $(0, y)$ in the z-plane is $(-y^2, 0)$. Hence as $(0, y)$ travels downward to the origin along the y axis, its image moves to the right along the negative u axis and reaches the origin in the w-plane. Then, since the image of a point $(x, 0)$ is $(x^2, 0)$, that image moves to the right from the origin along the u axis as $(x, 0)$ moves to the right from the origin along the x axis. The image of the upper branch of the hyperbola $xy = 1$ is, of course, the horizontal line $v = 2$. Evidently, the closed region

$$D = \{(x, y): x \geq 0, y \geq 0, xy \leq 1\}$$

is mapped onto the closed strip $D' = \{(u, v): 0 \leq v \leq 2\}$, as indicated in Fig. 2-3.

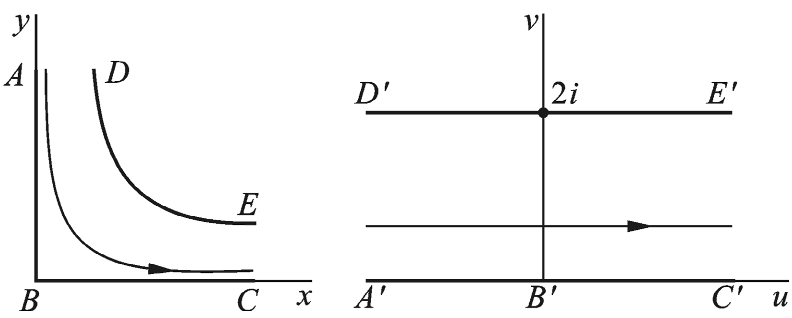

Fig. 2-3

Our last example illustrates how polar coordinates can be used in analyzing certain mappings.

Example 3. The mapping $w = z^2$ becomes $w = r^2 e^{i2\theta}$ when $z = re^{i\theta}$. Hence, if we write $w = \rho e^{i\phi}$, then we have $\rho e^{i\phi} = r^2 e^{i2\theta}$; and Propositon 1.8.1(2) tells us that

$$\rho = r^2 \quad \text{and} \quad \phi = 2\theta + 2k\pi ,$$

where k has one of the values $k = 0, \pm 1, \pm 2, \ldots$. Evidently, then, the image of any nonzero point z is found by squring the modulus of z and doubling a value of $\operatorname{Arg} z$.

Observe that points $z = r_0 e^{i\theta}$ on a circler $r = r_0$ are transformed into points $w = r_0^2 e^{i2\theta}$ on the circle $\rho = r_0^2$. As a point on the first circle moves counterclockwise from the positive real axis to the positive imaginary axis, its image on the second circle moves counterclockwise from the positive real axis to the negative real axis (see Fig. 20). So, as all possible positive values of r_0 are chosen, the corresponding arcs in the z-plane and w-plane fill out the first quadrant and the upper half plane, respectively. The transformation $w = z^2$ is, then, a one to one mapping of the first quadrant $\{(r,\theta) : r \ge 0, 0 \le \theta \le \pi/2\}$ in the z-plane onto the upper-half plane $\{(\rho,\phi) : \rho \ge 0, 0 \le \phi \le \pi\}$ in the w-plane, as indicated in Fig.2-4. The point $z = 0$ is, of course, mapped onto the point $w = 0$.

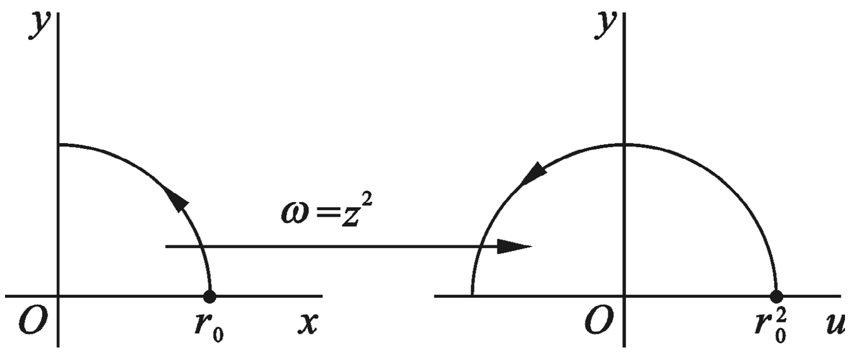

Fig. 2-4

The transformation $w = z^2$ also maps the upper half plane $\{(r,\theta) : r \ge 0, 0 \le \theta \le \pi\}$ onto the entire w-plane. However, in this case, the transformation is not one to one since both the positive and negative real axes in the z-plane are mapped onto the positive real axis in the w-plane.

When n is a positive integer greater than 2, various mapping properties of the transform $w = z^n$, i.e., $\rho e^{i\phi} = r^n e^{in\theta}$, are similar to those of $w = z^2$. Such a transformation maps the entire z-plane onto the entire w-plane, where each nonzero point in the w-plane is the image of n distinct points in the z-plane. The circle $r = r_0$ is mapped onto the circle $\rho = r_0^n$; and the sector $\{(r,\theta) : 0 \le r \le r_0, 0 \le \theta \le 2\pi/n\}$ is mapped onto the disk $\rho \le r_0^n$, but not in a one to one manner.