Chapter Ⅵ

Residues and Poles

The Cauchy integral theorem (Sec. 4.9) states that if a function is analytic at all points interior to and on a simple closed path C, then the value of the integral of the function around that path is zero. If, however, the function fails to be analytic at a finite number of points interior to C, there is, as we shall see in this chapter, a specific number, called a residue, which each of those points contributes to the value of the integral. We develop here the theory of residues; and, in Chapter Ⅶ, we shall give some applications of the theory.

§6.1. Residues

Recall (Sec. 2.13) that a point z_0 is a *singular point* of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.

Definition 6.1.1. A singular point z_0 of a function f is said to be *isolated* if there is an $\varepsilon > 0$ such that f is analytic for $0 < |z - z_0| < \varepsilon$.

Example 1. The function $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has the three isolated singular points $z = 0$ and $z = \pm i$.

Example 2. The origin is a singular point of the principal branch (Sec. 3.3)

$$\log z = \ln r + i\theta (z = re^{i\theta})$$

of the logarithmic function. It is not, however, an *isolated* singular point since every deleted ε neighborhood of it contains points on the negative real axis (see Fig. 6-1) and the branch is not even defined there.

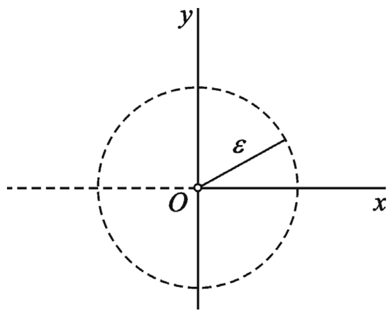

Fig. 6-1

Example 3. The function

$$f(z) = \frac{1}{\sin(\pi / z)}$$

has the singular point $z = 0$ and $z = 1/n(n = \pm 1, \pm 2, \ldots)$, all lying on the segment of the real axis from $z = -1$ to $z = 1$. Each singular point except $z = 0$ is isolated. The singular point $z = 0$ is not isolated because every deleted ε neighborhood of the origin contains other singular points of the function. More precisely, when a positive number ε is specified and m is any positive integer such that $m > 1/\varepsilon$, the fact that $0 < 1/m < \varepsilon$ means that the point $z = 1/m$ lies in the deleted ε neighborhood $0 < |z| < \varepsilon$ (Fig. 6-2)

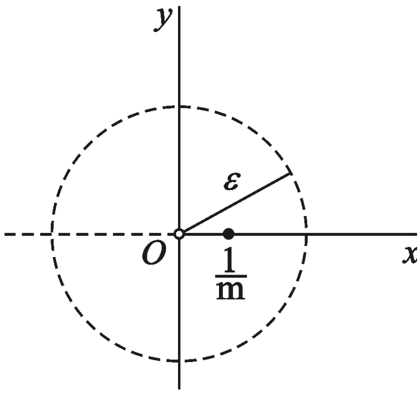

Fig. 6-2

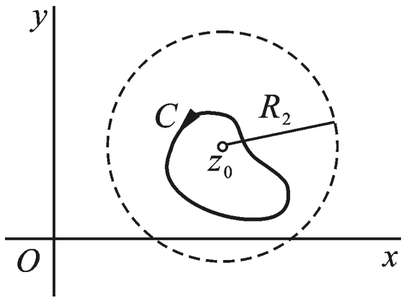

Fig. 6-3

When z_0 is an isolated singular point of a function f, there is a positive number R_2 such that f is analytic for $0 < |z - z_0| < R_2$. Consequently, $f(z)$ is represented by a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots \frac{b_n}{(z - z_0)^n} + \cdots \qquad (6.1.1)$$

where $0 < |z - z_0| < R_2$ and the coefficients a_n and b_n have certain integral representations (Sec. 5.4). In particular,

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}} \quad (n = 1, 2, \ldots)$$

where C is any positively oriented simple closed path around z_0 and lying in the punctured disk $0 < |z - z_0| < R_2$ (Fig. 6-3). When $n = 1$, this expression for b_n can be written

$$\int_C f(z)dz = 2\pi i b_1 . \qquad (6.1.2)$$

Definition 6.1.2. If z_0 is an isolated singular point of a function f, then the complex number b_1, which is the coefficient of $1/(z-z_0)$ in expansion (6.1.1), is called the *residue* of f at z_0 and denoted by $\operatorname*{Res}_{z=z_0} f(z)$, or $\mathrm{Res}(f(z), z_0)$. Thus,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_C f(z)dz \quad \text{and} \quad \int_C f(z)dz = 2\pi i \cdot \mathrm{Res}(f(z), z_0).$$

Equation (6.1.2) provides a powerful method for evaluating certain integrals around simple closed paths.

Example 4. Consider the integral

$$\int_C \frac{dz}{z(z-2)^4}, \qquad (6.1.3)$$

where C is the positively oriented circle $|z-2|=1$ (Fig. 6-4). Since the integrand is analytic everywhere in the finite plane except at the points $z=0$ and $z=2$, it has a Laurent series representation that is valid in the punctured disk $0<|z-2|<2$, also shown in Fig. 6-4.

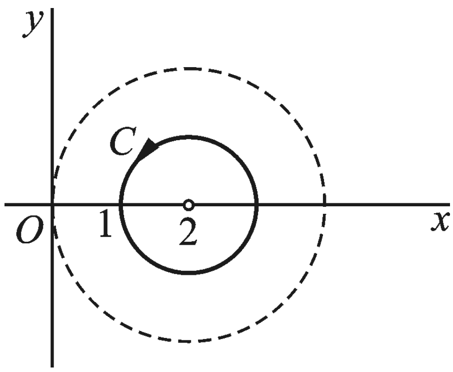

Fig. 6-4

Thus, according to equation (6.1.2), the value of integral (6.1.3) is $2\pi i$ times the residue of its integrand at $z=2$. To determine that residue, we recall (Sec. 5.3) the Maclaurin series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z|<1)$$

and use it to write

$$\begin{aligned}\frac{1}{z(z-2)^4} &= \frac{1}{(z-2)^4}\cdot\frac{1}{2+(z-2)} \\ &= \frac{1}{2(z-2)^4}\cdot\frac{1}{1-\left(-\dfrac{z-2}{2}\right)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^{n-4} \quad \left(0<|z-2|<2\right).\end{aligned}$$

In this Laurent series, the coefficient of $1/(z-2)$ is the desired residue, namely $-1/16$. Consequently,

$$\int_C \frac{dz}{z(z-2)^4} = 2\pi i\left(-\frac{1}{16}\right) = -\frac{\pi i}{8}. \qquad (6.1.4)$$

Example 5. Let us show that $\int_C \exp\left(\frac{1}{z^2}\right)dz = 0$, where C is the unit circle $|z| = 1$. Since $1/z^2$ is analytic everywhere except at the origin, so is the integrand. The isolated singular point $z = 0$ is interior to C; and, with the aid of the Maclaurin series (Sec. 5.3)

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (|z| < \infty),$$

one can write the Laurent series expansion

$$\exp\left(\frac{1}{z^2}\right) = 1 + \frac{1}{1!} \cdot \frac{1}{z^2} + \frac{1}{2!} \cdot \frac{1}{z^4} + \frac{1}{3!} \cdot \frac{1}{z^6} + \cdots \quad (0 < |z| < \infty).$$

Therefore, the residue of the integrand at its isolated singular point $z = 0$ is zero, and the value of the integral is established.

We are reminded in this example that, although the analyticity of a function within and on a simple closed path C is a sufficient condition for the value of the integral around C to be zero, it is not a necessary condition.