§8.4. Affine Transformations

To study the mapping

$$w = Az, \tag{8.4.1}$$

where A is a nonzero complex constant and $z \neq 0$, we write A and z in exponential form:

$$A = ae^{i\alpha}, \quad z = re^{i\theta}.$$

Then

$$w = (ar)e^{i(\alpha+\theta)}, \tag{8.4.2}$$

and we see from equation (8.4.2) that transformation (8.4.1) expands or contracts the radius vector representing z by the factor $a = |A|$ and rotates it through an angle $\alpha = \arg A$ abount the origin. The image of a given region is, therefore, geometrically similar to that region. In this sence, we call (8.4.1) a *similarity*.

The mapping

$$w = z + B, \tag{8.4.3}$$

where B is any complex constant, is a *translation* by means of the vector representing B. That is, if we write

$$w = u + iv, \quad z = x = iy, \text{ and } \quad B = b_1 + ib_2,$$

then the image of any point (x, y) in the z plane is the point

$$(u, v) = (x + b_1, y + b_2) \tag{8.4.4}$$

in the w plane. Since each point in any given region of the z plane is mapped into the w plane in this manner, the image region is geometrically congruent to the original one.

Generally, the transformation

$$w = Az + B \quad (A \neq 0), \tag{8.4.5}$$

is called an *affine transformation,* which is a composition of the transformations

$$Z = Az \quad (A \neq 0) \quad \text{and} \quad w = Z + B.$$

Example. The mapping $w = (1+i)z + 2$ transforms the rectangular region shown in the z plane of Fig. 8-4 into the rectangular region shown in the w plane there. This is seen by writing it as a composition of the transformations

$$Z = (1+i)z \quad \text{and} \quad w = Z + 2.$$

Since $1 + i = \sqrt{2}\exp(i\pi/4)$, the first of these transformations is an expansion by the factor $\sqrt{2}$ and a rotation through the angle $\pi/4$. The second is a translation two units to the right.

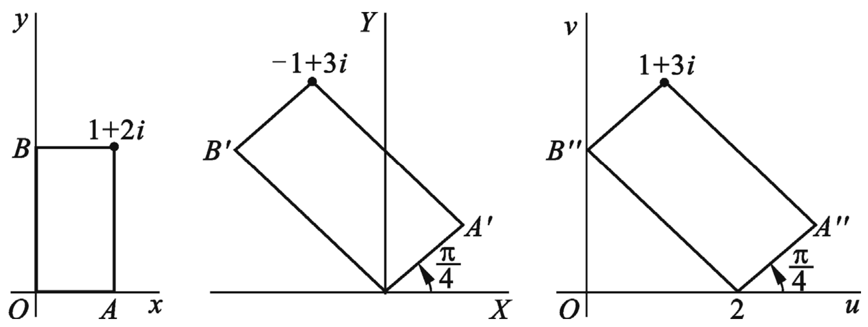

Fig. 8-4