§7.6. An Indentation Around a Branch Point

The example here involves the same indented path that was used in the example in the previous section. The indentation is, however, due to a branch point, rather than an isolated singularity.

Example. The integration formula

$$\int_0^{\infty} \frac{\ln x}{(x^2+4)^2}\,dx = \frac{\pi}{32}(\ln 2 - 1) \qquad (7.6.1)$$

can be derived by considering the branch

$$f(z) = \frac{\log z}{(z^2+4)^2} \qquad \left(|z|>0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

of the multiple-valued function $(\mathrm{Log} z)/(z^2+4)^2$. This branch, whose branch cut consists of the origin and the negative imaginary axis, is analytic everywhere in the indicated domain except at the point $z = 2i$. In order that the isolated singularity $2i$ always be inside the closed path, we require that $\rho < 2 < R$. See Fig. 7-9, where the isolated singularity and the branch point $z = 0$ are shown and where the same labels L_1, L_2, C_ρ, and C_R as in Fig. 7-9 are used. According to Cauchy's residue theorem,

$$\int_{L_1} f(z)dz + \int_{C_R} f(z)dz + \int_{L_2} f(z)dz + \int_{C_\rho} f(z)dz = 2\pi i \operatorname*{Res}_{z=2i} f(z).$$

That is,

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\pi i \operatorname*{Res}_{z=2i} f(z) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz. \qquad (7.6.2)$$

Since

$$f(z) = \frac{\ln r + i\theta}{(r^2 e^{i2\theta}+4)^2} \qquad (z = re^{i\theta}),$$

the parametric representations

$$z = x(\rho \le x \le R) \quad \text{and} \quad z = x(-R \le x \le -\rho) \qquad (7.6.3)$$

for the legs L_1 and $-L_2$ can be used to write the left-hand side of equation (7.6.2) as

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = \int_\rho^R \frac{\ln x}{(x^2+4)^2}dx + \int_\rho^R \frac{\ln x + i\pi}{(x^2+4)^2}dx$$

since $r = |x|, \theta = \pi$ when $z \in L_2$.

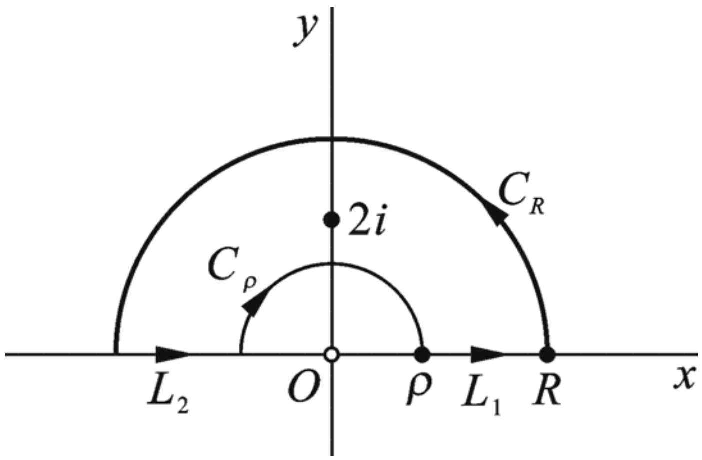

Fig. 7-9

Also, since

$$f(z)=\frac{\phi(z)}{(z-2i)^2} \quad \text{where} \quad \phi(z)=\frac{\log z}{(z+2i)^2},$$

the singularity $z=2i$ of $f(z)$ is a pole of order 2, with residue

$$\phi'(2i)=\frac{\pi}{64}+i\frac{1-\ln 2}{32}.$$

Equation (7.6.2) thus becomes

$$\begin{aligned}&2\int_\rho^R \frac{\ln x}{(x^2+4)^2}dx+i\pi\int_\rho^R \frac{dx}{(x^2+4)^2}\\ &=\frac{\pi}{16}(\ln 2-1)+i\frac{\pi^2}{32}-\int_{C_\rho} f(z)dz-\int_{C_R} f(z)dz;\end{aligned} \tag{7.6.4}$$

and, by equating the real parts on each side here, we find that

$$2\int_\rho^R \frac{\ln r}{(r^2+4)^2}dr=\frac{\pi}{16}(\ln 2-1)-\operatorname{Re}\int_{C_\rho} f(z)dz-\operatorname{Re}\int_{C_R} f(z)dz. \tag{7.6.5}$$

It remains only to show that

$$\lim_{\rho\to 0}\operatorname{Re}\int_{C_\rho} f(z)dz=0 \quad \text{and} \quad \lim_{R\to\infty}\operatorname{Re}\int_{C_R} f(z)dz=0. \tag{7.6.6}$$

Limits (7.6.6) are established as follows. First, we note that if $0<\rho<1$ and $z=\rho e^{i\theta}$ is a point on C_ρ, then

$$|\log z|=|\ln\rho+i\theta|\le|\ln\rho|+|i\theta|\le-\ln\rho+\pi$$

and

$$|z^2+4|\ge||z^2-4|=4-\rho^2.$$

As a consequence.

$$\left|\operatorname{Re}\int_{C_\rho} f(z)dz\right|\le\left|\int_{C_\rho} f(z)dz\right|\le\frac{-\ln\rho+\pi}{(4-\rho^2)^2}\pi\rho=\pi\frac{\pi\rho-\rho\ln\rho}{(4-\rho^2)^2};$$

and, by Hospital's rule, the product $\rho\ln\rho$ in the numerator on the far right here tends to 0 as ρ tends to 0. So the first of limits (7.6.6) clearly holds. Likewise, by writing

$$\left|\operatorname{Re}\int_{C_R} f(z)dz\right|\le\left|\int_{C_R} f(z)dz\right|\le\frac{\ln R+\pi}{(R^2-4)^2}\pi R=\pi\frac{\dfrac{\pi}{R}+\dfrac{\ln R}{R}}{\left(R-\dfrac{4}{R}\right)^2}$$

and using Hospital's rule to show that the quotient $(\ln R)/R$ tends to 0 as R tends to ∞, we obtain the second of limits (7.6.6).

Note how another integration formula, namely

$$\int_0^\infty \frac{dx}{(x^2+4)^2}=\frac{\pi}{32}, \tag{7.6.7}$$

follows by equating imaginary, rather than real, parts on each side of equation (7.6.4):

$$\pi\int_\rho^R \frac{dr}{(r^2+4)^2}=\frac{\pi^2}{32}-\operatorname{Im}\int_{C_\rho} f(z)dz-\operatorname{Im}\int_{C_R} f(z)dz. \tag{7.6.8}$$

Formula (7.6.7) is then obtained by letting ρ and R tend to 0 and ∞, respectively, since

$$\left|\operatorname{Im}\int_{C_\rho} f(z)dz\right|\le\left|\int_{C_\rho} f(z)dz\right| \quad \text{and} \quad \left|\operatorname{Im}\int_{C_R} f(z)dz\right|\le\left|\int_{C_R} f(z)dz\right|.$$