§4.5. Examples

The purpose of this section is to provide examples of the definition in Sec. 4.4 of path integrals and to illustrate various properties that were mentioned there. We defer development of the concept of antiderivatives of the integrands $f(z)$ in path integrals until Sec. 4.7.

Example 1. Let us find the value of the integral

$$I = \int_C \bar{z} dz \tag{4.5.1}$$

when C is the right-hand half (Fig. 4-7)

$$z = 2e^{i\theta} \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$$

of the circle $|z| = 2$, from $z = -2i$ to $z = 2i$. According to definition (4.4.2), Sec.4.4,

$$I = \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}} (2e^{i\theta})' d\theta = \int_{-\pi/2}^{\pi/2} 2e^{-i\theta} 2ie^{i\theta} d\theta = 4i \int_{-\pi/2}^{\pi/2} d\theta = 4\pi i .$$

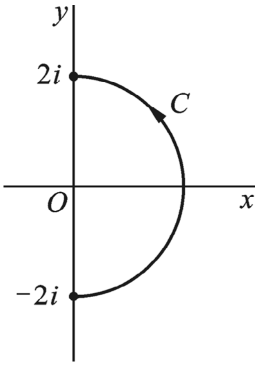

Fig. 4-7

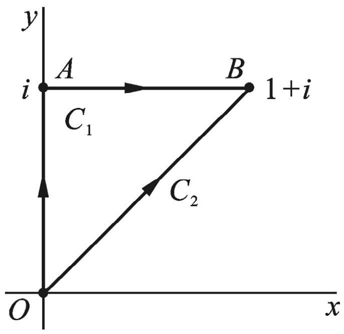

Fig. 4-8

Note that when a point z is on the circle $|z| = 2$, it follows that $z\bar{z} = 4$, or $\bar{z} = 4/z$. Hence the result $I = 4\pi i$ can also be written

$$\int_C \frac{dz}{z} = \int_C \frac{1}{4} \bar{z} dz = \frac{1}{4} \int_C \bar{z} dz = \pi i . \tag{4.5.2}$$

Example 2. Let $f(z) = y - x - i3x^2 (z = x + iy)$, C_1 denote the path OAB shown in Fig. 4-8, and C_2 denote the segment OB of the line $y = x$. Find the integerals of f along C_1 and C_2.

Solution. From the formula (4.4.6), we get

$$\int_{C_1} f(z) dz = \int_{OA} f(z) dz + \int_{AB} f(z) dz . \tag{4.5.3}$$

Since the leg OA may be represented parametrically as $z = 0 + iy (0 \le y \le 1)$, by definition (4.4.2) we compute that

$$\int_{OA} f(z) dz = \int_0^1 y i dy = i \int_0^1 y dy = \frac{i}{2} .$$

On the leg $AB : z = x + i (0 \le x \le 1)$, we have

$$\int_{AB} f(z) dz = \int_0^1 (1 - x - i3x^2) \cdot 1 dx = \int_0^1 (1 - x) dx - 3i \int_0^1 x^2 dx = \frac{1}{2} - i .$$

In view of equation (4.5.3), we now see that

$$\int_{C_1} f(z) dz = \frac{1 - i}{2} . \tag{4.5.4}$$

Since C_2 has a parametric representation $z = x + ix (0 \le x \le 1)$, we see that

$$\int_{C_2} f(z)dz = \int_0^1 [-i3x^2(1+i)]dx = 3(1-i)\int_0^1 x^2 dx = 1 - i. \qquad (4.5.5)$$

Evidently, then, the integrals of f along the two paths C_1 and C_2 have different values even though those paths have the same initial and the same final points.

Observe how it follows that the integral of f over the simple closed path $C_1 - C_2$, *i.e.*, $OABO$, has the *nonzero value*

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = \frac{-1+i}{2}.$$

Example 3. We begin here by letting C denote an arbitrary smooth arc

$$z = z(t) \quad (a \le t \le b)$$

from a fixed point z_1 to a fixed point z_2 (Fig. 4-9).

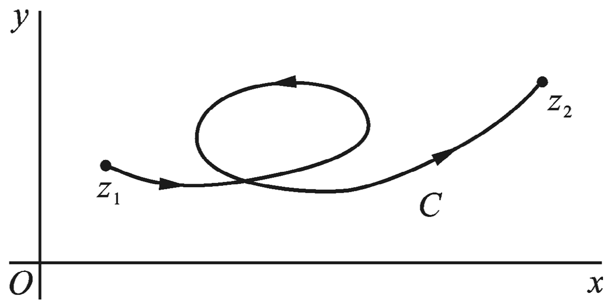

Fig. 4-9

In order to evaluate the integral

$$I = \int_C z dz = \int_a^b z(t)z'(t)dt,$$

we note that, according to Exercise $1(b)$, Sec. 4.2,

$$\frac{d}{dt}\frac{[z(t)]^2}{2} = z(t)z'(t).$$

It follows from the formula (4.2.4) that

$$I = \left.\frac{[z(t)]^2}{2}\right|_a^b = \frac{[z(b)]^2 - [z(a)]^2}{2}.$$

But $z(b) = z_2$ and $z(a) = z_1$; and so

$$I = (z_2^2 - z_1^2)/2.$$

This shows that the value of I depends only on the end points of C, and is not independent of the arc that is taken. Thus, we may write

$$\int_{z_1}^{z_2} z dz = \frac{z_2^2 - z_1^2}{2}. \qquad (4.5.6)$$

(Compare Example 2, where the value of an integral from one fixed point to another depended on the path that was taken.)

Expression (4.5.6) is also valid when C is a path that is not necessarily smooth since a path consists of a finite number of smooth arcs $C_k (k = 1,2,\ldots,n)$, joined end to end. More precisely, suppose that each C_k extends from z_k to z_{k+1}. Then by (4.5.6) we have

$$\int_C zdz = \sum_{k=1}^{n} \int_{C_k} zdz = \sum_{k=1}^{n} \frac{z_{k+1}^2 - z_k^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}, \tag{4.5.7}$$

z_1 being the initial point of C and z_{n+1} its final point.

It follows from expression (4.5.7) that the integral of the function $f(z) = z$ around each closed path in the plane has value zero. (Once again, compare Example 2, where the value of the integral of a given function around a certain closed path was not zero.) The question of predicting when an integral around a closed path has value zero will be discussed in Sec. 4.7, 4.9, and 4.11.

Example 4. Let C denote the semicircular path

$$z = 3e^{i\theta} \quad (0 \le \theta \le \pi)$$

from the point $z = 3$ to the point $z = -3$ (Fig. 4-10). Although the branch (Sec.3.3)

$$f(z) = \sqrt{r}e^{i\theta/2} \quad (r > 0, 0 < \theta < 2\pi) \tag{4.5.8}$$

of the multiple-valued function $z^{1/2}$ is not defined at the initial point $z = 3$ of the path C, the integral

$$I = \int_C f(z)dz \tag{4.5.9}$$

of that branch nevertheless exists.

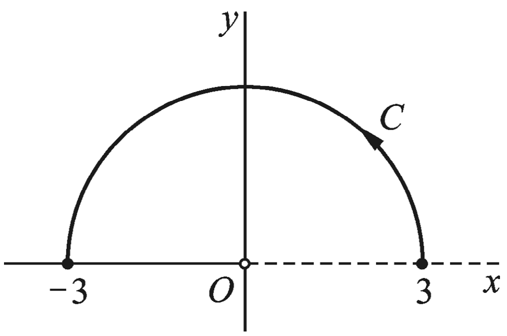

Fig. 4-10

To see that this is so, we observe that

$$f[z(\theta)] = \sqrt{3}e^{i\theta/2} = \sqrt{3}\cos\frac{\theta}{2} + i\sqrt{3}\sin\frac{\theta}{2} \to \sqrt{3} \quad (\theta \to 0^+)$$

Hence f is continuous on C whenever its value at $z = 3$ is defined as $\sqrt{3}$. Consequently,

$$I = \int_0^{\pi} \sqrt{3}e^{i\theta/2} 3ie^{i\theta} d\theta = 3\sqrt{3}i \int_0^{\pi} e^{i3\theta/2} d\theta;$$

and

$$\int_0^{\pi} e^{i3\theta/2} d\theta = \frac{2}{3i} e^{i3\theta/2} \Big|_0^{\pi} = -\frac{2}{3i}(1+i).$$

Finally, we obtain $I = -2\sqrt{3}(1+i)$.