§1.8. Roots of Complex Numbers

Consider now a point $z = re^{i\theta}$, lying on a circle centered at the origin with radius r (Fig. 1-10). As θ is increased, z moves around the circle in the counterclockwise direction. In particular, when θ is increased by 2π, we arrive at the original point; and the same is true when θ is decreased by 2π.

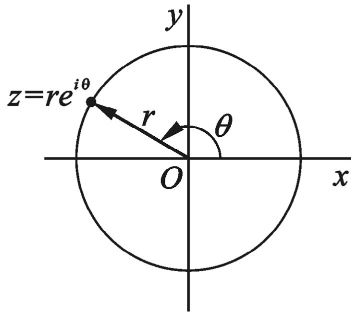

Fig. 1-10

1.Proposition

(1) *Let* α *be a real number, then* $e^{i\alpha} = 1 \Leftrightarrow \alpha = 2k\pi$ *for some integer* k.

(2) *Two nonzero complex numbers* $z_1 = r_1 e^{i\theta_1}$ *and* $z_2 = r_2 e^{i\theta_2}$ *are equal if and only if* $r_1 = r_2$ *and* $\theta_1 = \theta_2 + 2k\pi$ *for some integer* k.

2.Definition of roots of complex number

Let $z_0 = r_0 e^{i\theta_0}$ be any nonzero complex number and $n \geq 2$ be a positive integer. A complex number $z = re^{i\theta}$ is said to be an n th *root of* z_0 if it satisfies the equation $z^n = z_0$.

Let $z = re^{i\theta}$ be an n th root of $z_0 = r_0 e^{i\theta_0}$, then $r^n e^{in\theta} = r_0 e^{i\theta_0}$,so $r = \sqrt[n]{r_0}$,

$$\theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n}.$$

Consequently, $z = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right]$ for some integer k.

On the other hand, it is obvious that the numbers

$$c_k = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right] (k = 0, \pm 1, \pm 2, \ldots). \qquad (1.8.1)$$

are the nth roots of z_0 ,so

$$z_0^{1/n} = \sqrt[n]{r_0} \exp\left[\frac{i(\theta_0 + 2k\pi)}{n} \right] (k = 0, 1, 2, \ldots, n-1). \qquad (1.8.2)$$

Examples in the next section serve to illustrate this method for finding roots of complex numbers.