§1.4. Moduli

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z (Sec. 1.1) in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 1-2, the number $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.

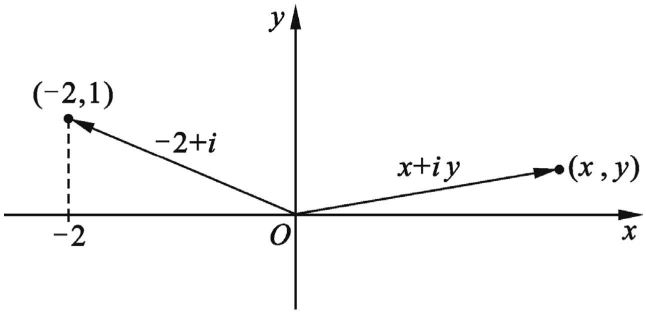

Fig. 1-2

According to the definition of the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $z_1 + z_2$ may be obtained vectorially as shown in Fig. 1-3.

The difference $z_1 - z_2 = z_1 + (-z_2)$ corresponds to the sum of the vectors z_1 and $-z_2$ (Fig. 1-4).

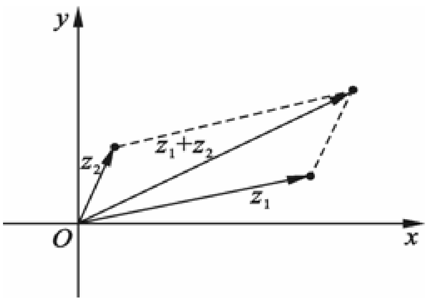

Fig. 1-3

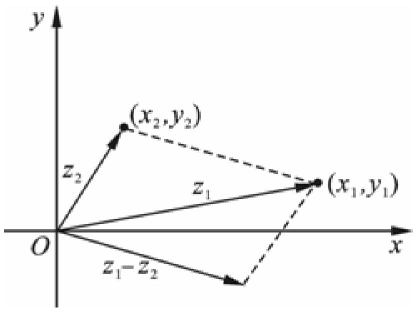

Fig. 1-4

1. Modulus

The *modulus*, or *absolute value*, of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

$$|z| = \sqrt{x^2 + y^2}\,. \tag{1.4.1}$$

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the vector representing z. It reduces to the usual absolute value in the real number *system* when $y = 0$.

2.Distance of complex numbers

The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is defined by

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} .$$

The complex numbers z corresponding to the points lying on the circle with center z_0 and radius R thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of these points simply as the circle $|z - z_0| = R$, denoted by $C(z_0, R)$.

Example 2. The equation $|z - 1 + 3i| = 2$ represents the circle whose center is the point $z_0 = (1, -3)$ and whose radius is $R = 2$.

3.Relationshps of z, $\operatorname{Re} z$ and $\operatorname{Im} z$

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 . \tag{1.4.2}$$

$$\operatorname{Re} z \le |\operatorname{Re} z| \le |z| \text{ and } \operatorname{Im} z \le |\operatorname{Im} z| \le |z| . \tag{1.4.3}$$

4.Triangle inequality

$$|z_1 \pm z_2| \le |z_1| + |z_2| , \tag{1.4.4}$$

$$|z_1 \pm z_2| \ge \big||z_1| - |z_2|\big| . \tag{1.4.5}$$

Example 3. If a point z lies on the unit circle $|z| = 1$ about the origin, then

$$|z - 2| \le |z| + 2 = 3$$

and

$$|z - 2| \ge \big||z| - 2\big| = 1 .$$

The triangle inequality (1.4.4) can be generalized by means of mathematical induction to sums involving any finite number of terms:

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n| \quad (n = 2, 3, \ldots) . \tag{1.46}$$