§2.6. Limits Involving the Point at Infinity

1.Definition of extended complex plane

It is sometimes convenient to include with the complex plane the *point at infinity*, denoted by ∞, and to use limits involving it. The complex plane together with this point is called the *extended* complex plane and denoted by $\mathbf{C}_\infty$. Thus, $\mathbf{C}_\infty = \mathbf{C} \bigcup \{\infty\}$. To visualize the point at infinity, one can think of the complex plane as passing through the equator of a unit sphere $S(0,1)$ centered at the point $z = 0$ (Fig. 2-11).

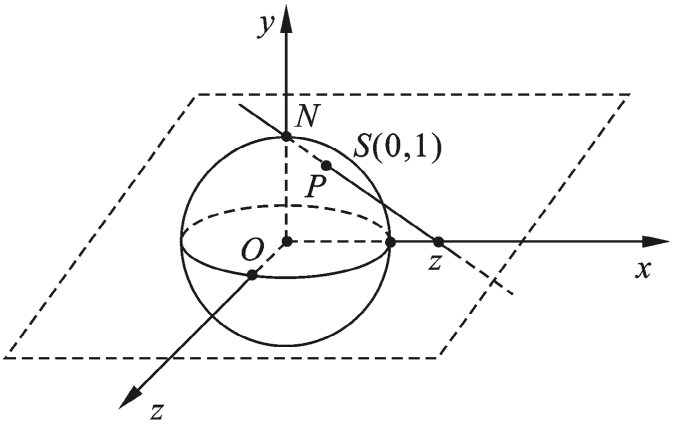

Fig. 2-11

2. Definition of a stereographic projection

To each point z in the plane there corresponds exactly one point P on the surface of the sphere. The point P is determined by the intersection of the line through the point z and the north N of the sphere with that surface. In like manner, to each point P on the surface of the sphere, other than the north pole N, there corresponds exactly one point z in the plane. By letting the point N of the sphere correspond to the point at infinity, we obtain a one to one correspondence between the points of the extended complex plane. The sphere is known as the *Riemann sphere*, and the correspondence is called a ***stereographic projection***. Under this correspondence, we have $N \leftrightarrow \infty, P \leftrightarrow z$.

3. Definition of a neighborhood of ∞

Observe that the exterior of the unit circle centered at the origin in the complex plane corresponds to the upper hemisphere with the equator and point N deleted. Moreover, for each small positive number ε, those points in the complex plane exterior to the circle $|z| = 1/\varepsilon$ correspond to points on the sphere close to N. The set $\{z : |z| > 1/\varepsilon\}$ is called an ε *-neighborhood*, or *neighborhood*, of ∞.

Let us agree that, in referring to a point z, we mean a point in the finite plane. Hereafter, when the point at infinity is to be considered, it will be specifically mentioned.

A meaning is now readily given to the statement

$$\lim_{z \to z_0} f(z) = w_0$$

when z_0, w_0 are extended complex numbers, *i.e.*, elements of the extended complex plane $\mathbf{C}_\infty$. In the definition of limit in Sec. 2.4, we simply replace the appropriate neighborhoods of z_0 and w_0 by neighborhoods of ∞. For example,

$\lim_{z \to z_0} f(z) = \infty$ if $\forall M > 0, \exists \delta > 0$ such that

$$|f(z)| > M \quad \text{if} \quad 0 < |z - z_0| < \delta,$$

and $\lim_{z\to\infty} f(z) = w_0$ if $\forall \varepsilon, \exists M > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad |z| > M.$$

Theorem 2.6.1. *If z_0 and w_0 are points in the z-plane and w-plane, respectively, then*

$$\lim_{z\to z_0} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z\to z_0} \frac{1}{f(z)} = 0 \tag{2.6.1}$$

and

$$\lim_{z\to\infty} f(z) = w_0 \quad \textit{if and only if} \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0. \tag{2.6.2}$$

Moreover,

$$\lim_{z\to\infty} f(z) = \infty \quad \textit{if and only if} \quad \lim_{z\to 0} \frac{1}{f(1/z)} = 0. \tag{2.6.3}$$

Example. Observe that

$$\lim_{z\to -1} \frac{iz+3}{z+1} = \infty \quad \text{since} \quad \lim_{z\to -1} \frac{z+1}{iz+3} = 0$$

and

$$\lim_{z\to\infty} \frac{2z+i}{z+1} = 2 \quad \text{since} \quad \lim_{z\to 0} \frac{(2/z)+i}{(1/z)+1} = \lim_{z\to 0} \frac{2+iz}{1+z} = 2.$$

Furthermore,

$$\lim_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty \quad \text{since} \quad \lim_{z\to 0} \frac{(1/z^2)+1}{(2/z^3)-1} = \lim_{z\to 0} \frac{z+z^3}{2-z^3} = 0.$$