## §4.4. Path Integrals

We turn now to integrals of complex-valued functions $f$ of the complex variable $z$. Such an integral is defined in terms of the values $f(z)$ along a given path $C$ extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral; and its value depends, in general, on the path $C$ as well as on the function $f$. It is written

$$\int_C f(z)dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)dz,$$

the latter notation often being used when the value of the integral is independent of the choice of the path taken between two fixed end points. While the integral may be defined directly as the limit of a sum, we choose to define it in terms of a definite integral of the type introduced in Sec.4.2.

**Definition 4.4.1.** Suppose that the equation

$$z = z(t) \quad (a \le t \le b) \tag{4.4.1}$$

represents a path $C$, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. Let the function $f$ be piecewise continuous on $C$. We define the *path integral* of $f$ along $C$ as follows:

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt. \tag{4.4.2}$$

Note that, since $C$ is a path, $z'(t)$ is also piecewise continuous on the interval $[a,b]$; and so the existence of integral (4.4.2) is ensured.

It follows immediately from definition (4.4.2) and properties of integrals of complex-valued functions $w(t)$ mentioned in Sec. 4.2 that

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz, \tag{4.4.3}$$

for any complex contents $z_0$, and

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz, \tag{4.4.4}$$

provided the integrals on the right-hand sides exist.

Associated with the path $C$ used in integral (4.4.2) is the path $-C$, consisting of the same set of points but with the order reversed so that the new path extends from the points $z_2$ to the point $z_1$ (Fig. 4-5).

The path $-C$ has parametric representation

$$z = z(-t) \quad (-b \le t \le -a);$$

and so, in view of Exercise $1(a)$, Sec. 4.2,

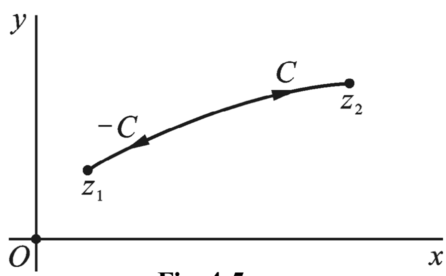


**Fig. 4-5**

$$\int_{-C} f(z)dz = \int_{-b}^{-a} f[z(-t)]\frac{d}{dt}z(-t)dt = -\int_{-b}^{-a} f[z(-t)]z'(-t)dt,$$

when $z'(-t)$ denotes the derivative of $z(t)$ with respect to $t$, evaluated at $-t$. Making the substitution $\tau = -t$ in this last integral and referring to Exercise $1(a)$, Sec. 4.3, we obtain the expression

$$\int_{-C} f(z)dz = -\int_a^b f[z(\tau)z'(\tau)d\tau ,$$

which is the same as

$$\int_{-C} f(z)dz = -\int_C f(z)dz . \qquad (4.4.5)$$

Consider now a path $C$ with representation (4.4.1) that consists of a path $C_1$ from $z_1$ to $z_2$ followed by a path $C_2$ from $z_2$ to $z_3$, the initial point of $C_2$ being the final point of $C_1$ (Fig. 4-6). There is a value $c$ of $t$ with $a < c < b$ such that $z(c) = z_2$.

Consequently, $C_1$ is represented by $z = z(t) \quad (a \le t \le c)$ and $C_2$ is represented by $z = z(t) \quad (c \le t \le b)$.

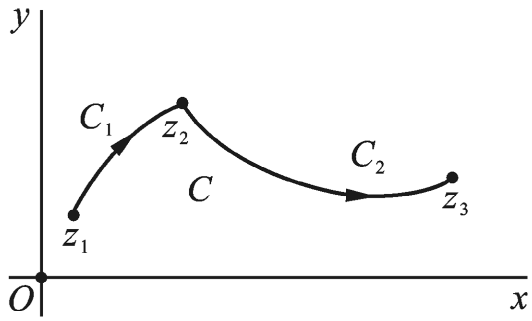


**Fig. 4-6**

Also, by a rule for integrals of functions $w(t)$ that was noted in Sec. 4.2,

$$\int_a^b f[z(t)]z'(t)dt = \int_a^c f[z(t)]z'(t)dt + \int_c^b f[z(t)]z'(t)dt .$$

Thus,

$$\int_C f(z) = dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz . \qquad (4.4.6)$$

Sometimes the path $C$ is called the *sum* of its legs $C_1$ and $C_2$ and is denoted by $C_1 + C_2$. The sum of two paths $C_1$ and $-C_2$ is well defined when $C_1$ and $C_2$ have the same final points, and it is written $C_1 - C_2$.

Definite integrals in calculus can be interpreted as areas, and they have other interpretations as well. Except in special case, no corresponding helpful interpretation, geometric or physical, is available for integrals in the complex plane.