§4.8. Examples

The following examples illustrate Theorem 4.7.1 in Sec. 4.7 and, in particular, the use of the extension (4.7.1) of the fundamental theorem of calculus in that section.

Example 1. The continuous function $f(z)=z^2$ has a primitive function $F(z)=z^3/3$ throughout the plane. Hence, by (4.7.4) we have

$$\int_0^{1+i} z^2 dz = \frac{z^3}{3}\Bigg|_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{2}{3}(-1+i)$$

for every path form $z=0$ to $z=1+i$.

Example 2. The function $f(z)=1/z^2$, which is continuous everywhere except at the origin, has a primitive function $F(z)=-1/z$ in the domain $|z|>0$, consisting of the entire plane with the origin deleted. Consequently, by Theorem 4.7.1, $\int_C \frac{dz}{z^2}=0$ whenever C is the positively oriented circle (Fig. 4-16)

$$z=2e^{i\theta} \quad (-\pi\le\theta\le\pi) \tag{4.8.1}$$

about the origin.

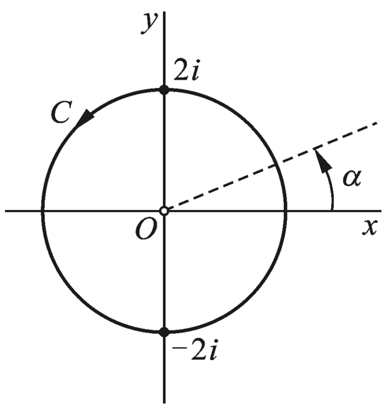

Fig. 4-16

Example 3. Let C_1 denote the right half

$$z=2e^{i\theta} \quad \left(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right) \tag{4.8.2}$$

of the circle C in Example 2. The principal branch

$$\log z=\ln r+i\theta \quad (z=re^{i\theta}, r>0, -\pi<\theta<\pi)$$

of the logarithmic function serves as a primitive function of the function $1/z$ in the evaluation of the integral of $1/z$ along C_1 (Fig. 4-17):

$$\begin{aligned}\int_{C_1}\frac{dz}{z} &= \int_{-2i}^{2i}\frac{dz}{z} = \log z\Big|_{-2i}^{2i} \\ &= \left(\ln 2+i\frac{\pi}{2}\right)-\left(\ln 2-i\frac{\pi}{2}\right) \\ &= \pi i.\end{aligned}$$

This integral was evaluated in another way in Example 1, Sec. 4.5, where representation (4.5.2) for the semicircle was used.

Next, let C_2 denote the left half

$$z = 2e^{i\theta} \quad \left(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right) \tag{4.8.3}$$

of the same circle C and consider the branch

$$L_0(z) = \ln r + i\theta \quad (z = re^{i\theta}, r > 0, 0 < \theta < 2\pi)$$

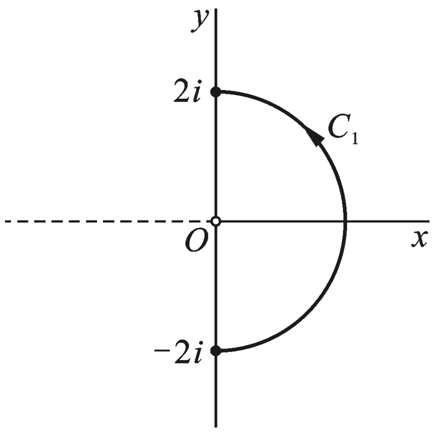

Fig. 4-17

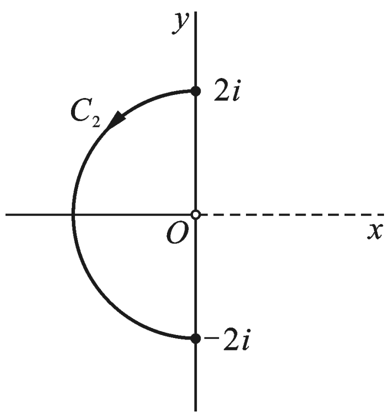

Fig. 4-18

of the logarithmic function (Fig. 4-18). One can write by (4.7.4) that

$$\int_{C_2} \frac{dz}{z} = \int_{2i}^{-2i} \frac{dz}{z} = L_0(z)\Big|_{2i}^{-2i} = L_0(-2i) - L_0(2i)$$

$$= \left(\ln 2 + i\frac{3\pi}{2}\right) - \left(\ln 2 + i\frac{\pi}{2}\right)$$

$$= \pi i.$$

The value of the integral of $1/z$ around the entire circle $C = C_1 + C_2$ is thus obtained:

$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = 2\pi i.$$

Example 4. Let us use a primitive function to evaluate the integral

$$\int_{C_1} f(z)dz, \tag{4.8.4}$$

where the integrand is the branch

$$f(z) = \sqrt{r}e^{i\theta/2} \quad (z = re^{i\theta}, r > 0, 0 < \theta < 2\pi) \tag{4.8.5}$$

of the square root function and C_1 is any path from $z = -3$ to $z = 3$ that, except for its end points, lies above the x axis (Fig. 4-19). We can define $f(3) = \sqrt{3}$ and the integrand becomes to be piecewise continuous on C_1 and the integral therefore exists. But another branch,

$$f_1(z) = \sqrt{r}e^{i\theta/2} \quad \left(z = re^{i\theta}, r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}\right),$$

is defined and continuous everywhere on C_1. The values of $f_1(z)$ at all points on

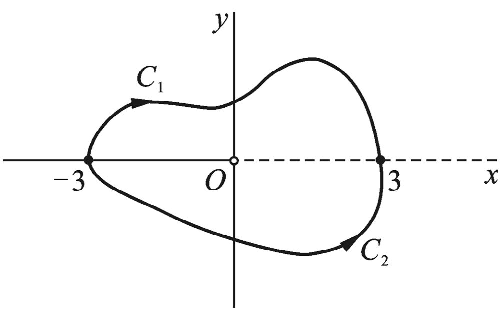

Fig. 4-19

C_1 except $z=3$ coincide with those of our integrand (4.8.5); so the integrand can be replaced by $f_1(z)$. Since a primitive function of $f_1(z)$ is the function

$$F_1(z)=\frac{2}{3}r\sqrt{r}e^{i3\theta/2}\quad\left(z=re^{i\theta},r>0,-\frac{\pi}{2}<\theta<\frac{3\pi}{2}\right),$$

we can now write

$$\int_{C_1} f(z)dz=\int_{-3}^{3} f_1(z)dz=F_1(z)\Big|_{-3}^{3}=2\sqrt{3}(e^{i0}-e^{i3\pi/2})=2\sqrt{3}(1+i).$$

(Compare Example 4 in Sec. 4.5.)

The integral

$$\int_{C_2} f(z)dz \tag{4.8.6}$$

of the function (4.8.5) over any path C_2 that extends from $z=-3$ to $z=3$ *below* the real axis can be evaluated in a similar way. In this case, we can replace the integrand by the branch

$$f_2(z)=\sqrt{r}e^{i\theta/2}\quad\left(z=re^{i\theta},r>0,\frac{\pi}{2}<\theta<\frac{5\pi}{2}\right),$$

whose values coincide with those of the integrand at $z=-3$ and at all points on C_2 below the real axis. This enables us to use a primitive function of $f_2(z)$ to evaluate integral (4.8.6). Details are left to the exercises.