§4.3. Paths

Integrals of complex-valued functions of a *complex* variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this section.

Definition 4.3.1. A set C of points $z=(x,y)$ in the complex plane is said to be an *arc* if there exist continuous functions x and y of the real parameter t on an interval $[a,b]$ such that

$$C=\{(x(t),y(t)):t\in[a,b]\}. \tag{4.3.1}$$

See Fig. 4-1 below.

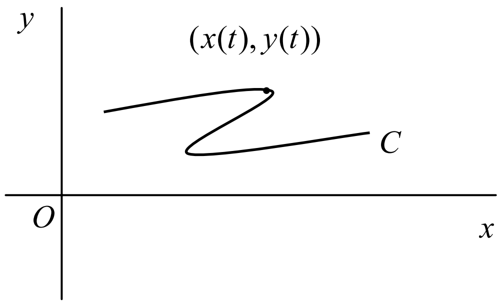

Fig. 4-1

This definition establishes a continuous mapping from the interval $[a,b]$ into the xy, or z - plane. If the image points are ordered according to increasing (*resp.* decreasing) values of t, then we call the arc to be *positively* (*resp. negatively*) *oriented,* shortly, *oriented*. It is convenient to describe the points of C by means of the equation

$$z=z(t)(a\le t\le b), \tag{4.3.2}$$

where

$$z(t)=x(t)+iy(t). \tag{4.3.3}$$

Definition 4.3.2. The oriented arc C given by (4.3.2) is called to be a *simple arc*, or a *Jordan arc*, if it does not cross itself; that is, for $a\le t_1\le t_2\le b$, we have

$$z(t_1)=z(t_2)\Rightarrow t_1=t_2, \text{or } \{t_1,t_2\}=\{a,b\}.$$

When the oriented arc C is simple such that $z(b)=z(a)$, we say that C is a *simple closed curve*, or a *closed Jordan curve*.

The geometric nature of a particular arc often suggests different notation for the parameter t in equation (4.3.2). This is, in fact, the case in the examples below.

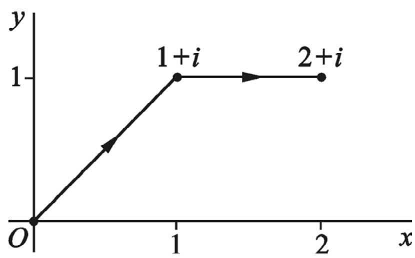

Fig. 4-2

Example 1. The *polygonal line* (Sec.1.10) defined by means of the function

$$z=\begin{cases} x+ix, & \text{when } 0\le x\le 1; \\ x+i, & \text{when } 1\le x\le 2, \end{cases} \tag{4.3.4}$$

consisting of a line segment from the point 0 to the point $1+i$ followed by one from the point $1+i$ to the point $2+i$ (Fig. 4-1) , is a simple arc.

Example 2. The unit circle

$$z=e^{i\theta}(0\le\theta\le 2\pi) \tag{4.3.5}$$

about the origin is a simple closed curve, oriented in the *counterclockwise direction*. So is the circle

$$z=z_0+R\cdot e^{i\theta}\quad(0\le\theta\le 2\pi), \tag{4.3.6}$$

centered at the point z_0 and with radius R (See Sec. 1.6, Fig. 4-3).

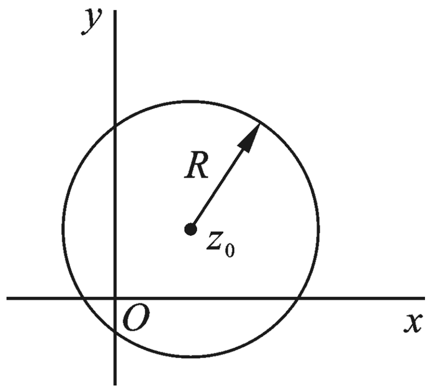

Fig. 4-3

The same set of points can make up different oriented arcs.

Example 3. The oriented arc

$$z = e^{-i\theta} \quad (0 \le \theta \le 2\pi), \tag{4.3.7}$$

oriented in the *clockwise direction*, is not the same as the oriented arc described by equation (4.3.5). The set is the same, but now the circle is traversed in the clockwise direction.

Example 4. The set of points on the arc

$$z = e^{2i\theta} \quad (0 \le \theta \le 2\pi) \tag{4.3.8}$$

is the same as the set making up the arcs (4.3.5) and (4.3.7). The arc here differs, however, from each of those arcs since the circle is traversed *twice* in the counterclockwise direction.

Suppose now that the components $x'(t)$ and $y'(t)$ of the derivative

$$z'(t) = x'(t) + iy'(t) \tag{4.3.9}$$

of the function (4.3.3), used to represent C, are continuous on the entire interval $[a,b]$. Such an arc C is often called a *differentiable arc*, and the real-valued function

$$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable over the interval $[\alpha,b]$. Moreover, according to the definition of arc-length in calculus, the length of C is the number

$$L = \int_a^b |z'(t)|\,dt. \tag{4.3.10}$$

If equation (4.3.2) represents a differentiable arc and if $z'(t) \ne 0$ anywhere in the interval $a < t < b$, then the *unit tangent vector*

$$\boldsymbol{T} = \frac{z'(t)}{|z'(t)|}$$

is well defined for all t in that open interval, with *angle of inclination* $\arg z'(t)$. Also, when $\boldsymbol{T}$ turns, it does so continuously as the parameter t varies over the entire interval (a,b). This expression for $\boldsymbol{T}$ is the one learned in calculus when $z(t)$ is interpreted as a *radius vector*.

Definition 4.3.3. An oriented arc

$$z = z(t)(a \le t \le b)$$

is called *smooth*, if the derivative $z'(t)$ is continuous on the closed interval $[a,b]$ and nonzero on the open interval (a,b).

Definition 4.3.4. A *path*, or *piecewise smooth arc*, is an oriented arc consisting of a finite number of smooth arcs joined end to end.

Hence, if equation (4.3.2) represents a path, then z is continuous, whereas its derivative z' is piecewise continuous. The polygonal line (4.3.4) is, for example, a path.

Definition 4.3.5. When only the initial and final values of z in a path C are the same, then we say that the path C is a *simple closed path.*

Examples of simple closed arcs are the circles (4.3.5) and (4.3.6), as well as the boundary of a triangle or a rectangle taken in a specific direction. The length of a path or a simple closed path is the sum of the lengths of the smooth arcs that make up the path.

Theorem 4.3.1(Jordan). *Any simple closed path C cuts the complex plane into two distinct domains, one of which is the inside of C that is bounded and denoted by $\text{ins}(C)$, the other is the outside of C that is unbounded and denoted by $\text{out}(C)$.*

So, the complex plane can be represented as

$$\mathbf{C} = \text{ins}(C) \cup C \cup \text{out}(C) \text{ (See Fig. 4-4)},$$

where C denotes the set $\{z(t) : a \le t \le b\}$ if the equation of C is $z = z(t)(a \le t \le b)$.

It will be convenient to accept this statement, as geometrically evident; the proof is not easy.

Fig. 4-4