## §8.3. Local Inverses

A transformation $w = f(z)$ that is conformal at a point $z_0$ has a local inverse there. That is, we have

**Theorem 8.3.1.** *If a transformation* $w = f(z)$ *is conformal at a* $z_0$ *and* $w_0 = f(z_0)$, *then there exists a unique transformation* $z = g(w)$, *which is defined and analytic in a neighborhood* $N$ *of* $w_0$, *such that* $g(w_0) = z_0$ *and* $f[g(w)] = w$ *for all pints* $w$ *in* $N$ *and the derivative of* $g(w)$ *is, moreover,*

$$g'(w) = \frac{1}{f'(z)}. \tag{8.3.1}$$

**Proof.** As noted in Sec. 8.1, the conformality of the transformation $w = f(z)$ at $z_0$ implies that there is a neighborhood $N(z_0, \delta)$ of $z_0$ throughout which $f$ is analytic and $f'(z) \neq 0$. Hence if we write

$$z = x + iy, \quad z_0 = x_0 + iy_0, \text{ and } f(z) = u(x, y) + iv(x, y),$$

we know that there is a neighborhood $N(z_0, \delta)$ of the point $(x_0, y_0)$ throughout which the functions $u(x, y)$ and $v(x, y)$ along with their partial derivatives of all orders, are continuous (see Sec. 4.13).

Now, the pair of equations

$$u = u(x, y), \quad v = v(x, y) \tag{8.3.2}$$

represents a transformation from the neighborhood $N(z_0, \delta)$ into the $uv$ plane. Moreover, the *Jacobian* determinant

$$J = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y = (u_x)^2 + (v_x)^2 = |f'(z)|^2$$

is nonzero on $N(z_0, \delta)$. If we write

$$u_0 = u(x_0, y_0) \text{ and } v_0 = v(x_0, y_0). \tag{8.3.3}$$

then from the theory of implicit functions in mathematical analysis, we know that there is a unique continuous transformation

$$x = x(u, v), \quad y = y(u, v), \tag{8.3.4}$$

defined on a neighborhood $N$ of the point $(u_0, v_0)$ such that

$$u = u(x(u,v), y(u,v)), \quad v = v(x(u,v), y(u,v)), \quad x_0 = x(u_0, v_0), \quad y_0 = y(u_0, v_0).$$

Also, the functions (8.3.4) have continuous first-order partial derivatives satisfying the equations

$$x_u = \frac{1}{J} v_y, \quad x_v = -\frac{1}{J} u_y, \quad y_u = -\frac{1}{J} v_x, \quad y_v = \frac{1}{J} u_x \tag{8.3.5}$$

throughout $N$.

If we write $w = u + iv$ and $w_0 = u_0 + iv_0$, as well as

$$g(w) = x(u, v) + iy(u, v), \tag{8.3.6}$$

the transformation $z = g(w)$ is evidently the local inverse of the original transformation $w = f(z)$ at $z_0$. Transformations (8.3.2) and (8.3.4) can be written

$$u + iv = u(x, y) + iv(x, y) \text{ and } x + iy = x(u, v) + iy(u, v);$$

and these last two equations are the same as

$$w = f(z) \text{ and } z = g(w),$$

where $g$ has the desired properties. From expression (8.3.5), we see that the function (8.3.4) have continuous partial derivatives and that the Cauchy-Riemann equations

$$x_u = y_v, x_v = -y_u$$

are satisfied in $N$. Thus, $g$ is analytic in $N$. Furthermore, we compute when $w = f(z)$,

$$g'(w) = x_u + iy_u = \frac{1}{J}(v_y - iv_x) = \frac{1}{J}(u_x - iv_x) = \frac{1}{u_x + iv_x} = \frac{1}{f'(z)}.$$

The proof is completed.

**Definition 8.3.1.** The function $g$ in Theorem 8.3.1 is called the local inverse of the function $f$ at $z_0$.

Clearly, unilateral function $f$ on a domain $D$ has an inverse function $f^{-1}: f(D) \to D$ that equals to the local inverse of $f$ at every point $z_0$. Thus, from Theorems 8.2.1, 8.2.3 and 8.3.1, we have

**Corollary 8.3.1.** *Every unilateral function* $f$ *on a domain* $D$ *has a unilateral inverse* $f^{-1}$ *on the domain* $f(D)$ *and*

$$f^{-1}(w) = \frac{1}{f'(z)}, \quad \forall w = f(z) \in f(D).$$

**Corollary 8.3.2.** *Every unilateral function* $f$ *on a domain* $D$ *is a homoemorphism from* $D$ *onto* $f(D)$, *i.e.* $f: D \to f(D), f^{-1}: f(D) \to D$ *are continuous.*

**Example 1.** We saw in Example 1, Sec. 9.1, that if $f(z) = e^z$, the mapping $w = f(z)$ is conformal everywhere in the $z$ plane and, in particular, at the point $z_0 = 2\pi i$. The image of this choice of $z_0$ is the point $w_0 = 1$. When points in the $w$ plane are expressed in the form $w = \rho \exp(i\phi)$, the local inverse at $z_0$ can be obtained by writing $g(w) = \log w$, where $\log w$ denotes the branch

$$\log w = \ln \rho + i\phi \qquad (\rho > 0, \pi < \theta < 3\pi)$$

of the logarithmic function, restricted to any neighborhood of $w_0$ that does not contain the origin. Observe that $g(1) = \ln 1 + i2\pi = 2\pi i$ and that, when $w$ is in the neighborhood,

$$f[g(w)] = \exp(\log w) = w.$$

Also,

$$g'(w) = \frac{d}{dw}\log w = \frac{1}{w} = \frac{1}{\exp z},$$

in accordance with equation (8.3.1).

Note that, if the point $z_0 = 0$ is chosen, one can use the principal branch

$$\log w = \ln \rho + i\phi \qquad (\rho > 0, -\pi < \theta < \pi)$$

of the logarithmic function to define $g$. In this case, $g(1) = 0$.