§4.10. Proof of Cauchy Integral Theorem

To complete the proof of the Cauchy integral theorem, we starte with several lemmas. Recall that a *simply connected domain* D is a domain such that every simple closed path within it encloses only points of D. Roughly speaking, a domain is simple connected if and only if it has no "holes".

Lemma 4.10.1. *Let* D *be a simple closed domain and* f *is analytic in* D*, then for every closed polygonal line* $C \subset D$*,*

$$\int_C f(z)dz = 0. \tag{4.10.1}$$

Proof. *Case* 1: Let $C \subset D$ be the boundary $\partial\Delta$ of a triangle Δ, see Fig. 4-20.

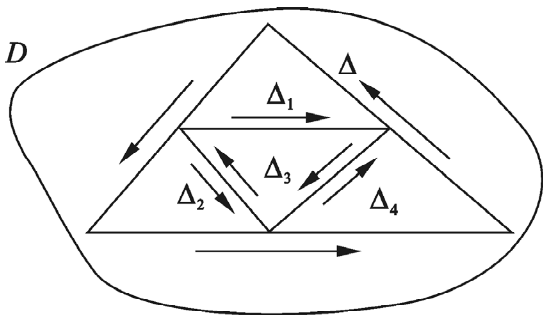

Fig. 4-20

Put $I = \left|\int_C f(z)dz\right|$. Joining the midpoints of the three sides of Δ divides the triangle Δ into four triangles $\Delta_1, \Delta_2, \Delta_3$ shown as in the figure. Clearly,

$$\int_C f(z)dz = \int_{\partial\Delta_1} f(z)dz + \int_{\partial\Delta_2} f(z)dz + \int_{\partial\Delta_3} f(z)dz + \int_{\partial\Delta_4} f(z)dz .$$

Thus, the modulus of one of the four integrals is larger than or equal to $\frac{I}{4}$, say

$$\left|\int_{\partial\Delta_1} f(z)dz\right| \ge \frac{I}{4}.$$

Similarly, joining the midpoints of the three sides of Δ_1 divides the triangle Δ_1 into four triangles in which there is at least one triangle, say Δ_2, with

$$\left|\int_{\partial\Delta_2} f(z)dz\right| \ge \frac{I}{4^2} \text{ and } \Delta_2 \subset \Delta_1.$$

Continuing this process gives a sequence $\{\Delta_n\}$ having the following properties:

$$\left|\int_{\partial\Delta_n} f(z)dz\right| \ge \frac{I}{4^n}, \Delta_{n+1} \subset \Delta_n, |\partial\Delta_n| = \frac{|\partial\Delta|}{2^n} (n = 1,2,\ldots). \tag{4.10.2}$$

where $|\partial\Delta|$ denotes the length of the curve $\partial\Delta$.

It is clear that the diameter $d(\Delta_n)$ of Δ_n goes to 0 as $n \to \infty$. Hence, by the finite covering theorem in calculus, we see that there exists a point

$$z_0 \in \Delta_n (n = 1,2,\ldots).$$

Since f is differentiable at z_0, for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon (\forall z \in N^\circ(z_0, \delta)).$$

Thus,

$$\left|f(z)-f(z_0)-f'(z_0)(z-z_0)\right|\le \varepsilon\left|z-z_0\right|(\forall z\in N(z_0,\delta)). \qquad (4.10.3)$$

Since $d(\Delta_n)\to 0(n\to\infty)$, there is an n such that $\Delta_n\subseteq N(z_0,\delta)$. By (4.10.3), we have

$$\left|f(z)-f(z_0)-f'(z_0)(z-z_0)\right|\le \varepsilon\left|z-z_0\right|\le \frac{|\partial\Delta|}{2^n}\varepsilon(\forall z\in\partial\Delta_n). \qquad (4.10.4)$$

From the fact that $\int_{\partial\Delta_n} 0dz=\int_{\partial\Delta_n} zdz=0$ and inequalities (4.10.4), we see that

$$\begin{aligned}\left|\int_{\partial\Delta_n} f(z)dz\right| &= \left|\int_{\partial\Delta_n}[f(z)-f(z_0)-f'(z_0)(z-z_0)]dz\right| \\ &\le \frac{|\partial\Delta|}{2^n}\varepsilon\cdot\frac{|\partial\Delta|}{2^n} \\ &= \frac{|\partial\Delta|^2}{4^n}\varepsilon.\end{aligned}$$

Thus, by (4.10.2) we get that $I\le|\partial\Delta|^2\varepsilon$. Since ε was arbitrary, we conclude that $I=0$.

Case 2: Let C be a closed polygonal line. Put $P=\overline{\mathrm{ins(C)}}$, then we can divided P into a finite number of triangles $\Delta_1,\Delta_2,\ldots,\Delta_n$, as in Fig. 4-21.

It is follows from Case 1 that

$$\int_C f(z)dz=\sum_{k=1}^{n}\int_{\partial\Delta_k} f(z)dz=0.$$

This completes the proof.

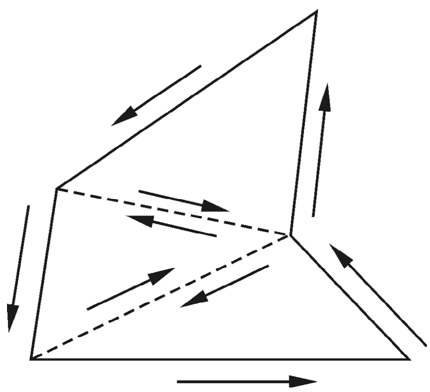

Fig. 4-21

A *convex domain* is a domain D such that for any two points $z_1,z_2\in D$, the line segment

$$[z_1,z_2]:=\{(1-t)z_1+tz_2: t\in[0,1]\}$$

is contained in D.

Clearly, every neighborhood is convex. And the annulus domain

$$A(z_0,R_1,R_2):=\{z: R_1<|z-z_0|<R_2\}$$

is not convex.

Lemma 4.10.2. *Let D be a convex domain and f be analytic in D, then f has a primitive function in D.*

Proof. Fixed a point $\alpha\in D$ and define

$$F(z)=\int_{\overline{\alpha z}} f(w)dw=(z-\alpha)\int_0^1 f((1-t)\alpha+tz)dt,$$

for all $z\in D$. We get a function $F: D\to\mathbf{C}$. We shall prove that

$$F'(z) = f(z), \forall z \in D. \tag{4.10.5}$$

Let $z, z_0 \in D$ with $z \neq z_0$. By Lemma 4.10.1 we know that

$$\int_{\overline{\alpha z_0}} f(w)dw + \int_{\overline{z_0 z}} f(w)dw + \int_{\overline{z z_0}} f(w)dw = 0.$$

That is, $F(z) - F(z_0) = \int_{\overline{z_0 z}} f(w)dw$. Thus,

$$F(z) - F(z_0) - (z - z_0)f(z_0) = \int_{\overline{z_0 z}} [f(w) - f(z_0)]dw.$$

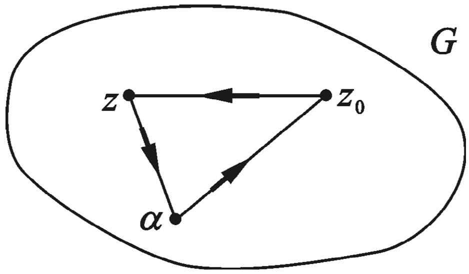

Fig. 4-22

Since f is continuous at z_0, $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$| f(w) - f(z_0) | < \varepsilon (\forall w \in N(z_0, \delta)).$$

Hence, for all $z \in N(z_0, \delta)$, we have

$$\left| F(z) - F(z_0) - (z - z_0)f(z_0) \right| = \left| \int_{\overline{z_0 z}} [f(w) - f(z_0)]dw \right|$$
$$\leq \varepsilon | z\text{-}z_0 |.$$

This shows that

$$F'(z_0) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0).$$

Therefore, the function F is a primitive function of f in D. The proof is completed.

The proof the Cauchy Inetgral Theorem. Suppose that f is analytic in $R = \overline{\text{ins(C)}}$, where C is a simple closed path. Thus, there is an open set $G \supset R$ such that f is analytic in it. Since the set C is a bounded closed set in the plane, we can find a finite number of open disks $D_1, D_2, \ldots, D_n \subset G$ such that

(1) $C \subset \bigcup_{k=1}^{n} D_k \subset G$;

(2) $C \cap D_k \cap D_{k+1} \neq \phi (k = 1, 2, \ldots, n)$, where $D_{n+1} = D_1$,

as in Fig. 4-23.

Take points $z_k \in C \cap D_k \cap D_{k+1} (k = 1, 2, \ldots, n)$ in the colockwise direction, as in Fig. 4-23. Then these points divide C into n simple paths $C_1, C_2, \ldots, C_n$ such that the initial and final points of C_k are z_{k-1}, z_k, respectively, where $z_0 = z_n$. Since each D_k is a convex domain and f is analytic in it, Lemma 4.10.2 yields that f has a primitive function F_k in D_k. It follows from Theorem 4.7.1 that

$$\int_{C_k} f(z)dz = \int_{\overline{z_{k-1} z_z}} f(z)dz (k = 1, 2, \ldots, n).$$

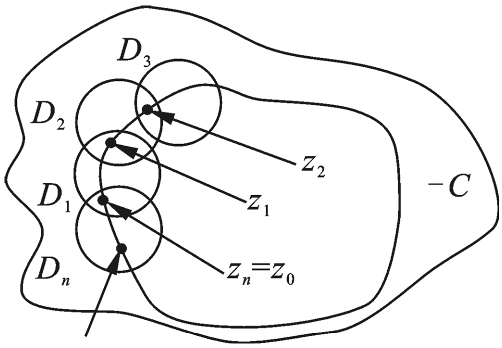

Fig. 4-23

Hence, from Lemma 4.10.1 we see that

$$\int_C f(z)dz = -\int_{-C} f(z)dz = -\sum_{k=1}^n \int_{C_k} f(z)dz = -\sum_{k=1}^n \int_{\overline{z_{k-1}z_z}} f(z)dz = 0,$$

since the curve

$$P = \sum_{k=1}^n \overrightarrow{z_{k-1}z_k}$$

is a polygonal line contained G. This completes the proof.