§7.7. Definite Integrals Involving Sine and Cosine

The method of residues is also useful in evaluating certain definite integrals of the type

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta\,. \tag{7.7.1}$$

The fact that θ varies from 0 to 2π suggests that we consider θ as an argument of a point z on the circle C centered at the origin. Hence we write

$$z = e^{i\theta} \quad (0 \le \theta \le 2\pi)\,. \tag{7.7.2}$$

Formally, then,

$$dz = ie^{i\theta}d\theta = izd\theta\,;$$

and the relations

$$\sin\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad d\theta = \frac{dz}{iz} \tag{7.7.3}$$

enable us to transform integral (7.7.1) into the contour integral

$$\int_C F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{iz} \tag{7.7.4}$$

of a function of z around the circle C in the positive direction. The original integral (7.7.1) is, of course, simply a parametric form of integral (7.7.4), in accordance with expression (7.7.2), Sec. 4.4. When the integrand of integral (7.7.4) is a rational function of z, we can evaluate that integral by means of Cauchy's residue theorem once the zeros of the polynomial in the denominator have been located and provided that none lie on C.

Example. Let us show that

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1 - a^2}} \quad (-1 < a < 1)\,. \tag{7.7.5}$$

This integration formula is clearly valid when $a = 0$, and we exclude that case in our derivation. With substitutions (7.7.3), the integral takes the form

$$\int_C \frac{2/a}{z^2 + (2i/a)z - 1}\,dz\,, \tag{7.7.6}$$

where C is the positively oriented circle $|z| = 1$. The quadratic formula reveals that the denominator of the integrand here has the pure imaginary zeros

$$z_1 = \left(\frac{-1 + \sqrt{1 - a^2}}{a}\right)i, \quad z_2 = \left(\frac{-1 - \sqrt{1 - a^2}}{a}\right)i\,.$$

So if $f(z)$ denotes the integrand, then

$$f(z) = \frac{2/a}{(z - z_1)(z - z_2)}\,.$$

Note that, because $|a| < 1$,

$$|z_2| = \frac{1 + \sqrt{1 - a^2}}{|a|} > 1\,.$$

Also, since $|z_1 z_2| = 1$, it follows that $|z_1| < 1$. Hence there are no singular points on C, and the only one interior to it is the point z_1. The corresponding residue B_1 is found by writing

$$f(z) = \frac{\phi(z)}{z - z_1} \quad \text{where} \quad \phi(z) = \frac{2/a}{z - z_2}\,.$$

This shows that z_1 is a simple pole and that

$$B_1 = \phi(z_1) = \frac{2/a}{z_1 - z_2} = \frac{1}{i\sqrt{1 - a^2}}\,.$$

Consequently,

$$\int_C \frac{2/a}{z^2 + (2i/a)z - 1} dz = 2\pi i B_1 = \frac{2\pi}{\sqrt{1-a^2}};$$

and integration formula (7.7.5) follows.

The method just illustrated applies equally well when the arguments of the sine and cosine are integral multiples of θ. One can use equation (7.7.2) to write, for example.

$$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{(e^{i\theta})^2 + (e^{i\theta})^{-2}}{2} = \frac{z^2 + z^{-2}}{2}.$$